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ON THE COMMUTANT OF TYPE I VON NEUMANN ALGEBRAS

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1. Introduction

The study of non-self-adjoint operator algebras on Hilbert space was begun in 1974 by Aveson (1). Recently, such algebras have been found to be of use in physics, in electrical engineering, and in general systems theory. Of particular interest to mathematicians are reflexive algebras with commutative lattices of invariant subspaces. One of the most important classes of such algebras is the sequence Alg$\mathcal{L}_2$, Alg$\mathcal{L}_4$, $\cdots$, Alg$\mathcal{L}_\infty$ of "tridiagonal" algebras, discovered by Gilfeather and Larson (4). We shall often disregard the distinction between an orthogonal projection and its range space. Let $\mathcal{L}$ be a family of orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then Alg$\mathcal{L}$ is an algebra containing I (I represents the identity operator acting on $\mathcal{H}$) and Alg$\mathcal{L}$ is closed in the weak operator topology.

In this paper, if Alg$\mathcal{L}$ is a von Neumann algebra, we want to find out what its type is. And we pursue a fragment of commutant theory leading to a revealing and useful description of type I von Neumann algebras. We will introduce the terminologies which are used in the above general introduction and in the general theorems of this paper. Let $\mathcal{C}$ be a subset of the class of all bounded operator acting on a Hilbert space $\mathcal{H}$. $\mathcal{C}$ is called self-adjoint if A^* is in $\mathcal{C}$ for every A in $\mathcal{C}$. If $\mathcal{C}$ is a vector space over $\mathbf{C}$ and if $\mathcal{C}$ is closed under the composition of maps, then $\mathcal{C}$ is called an algebra. $\mathcal{C}$ is called a self-adjoint algebra provided A^* is in $\mathcal{C}$ for every A in $\mathcal{C}$. Otherwise, $\mathcal{C}$ is called a non-self-adjoint algebra. $\mathcal{C}$ is a C^*-algebra if $\mathcal{C}$ is a self-adjoint algebra which is contains I and closed in the norm topology. $\mathcal{C}$ is a von Neumann algebra if $\mathcal{C}$ is a C^*-algebra which is closed in the weak operator topology. For any subset $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, we shall denote by $\mathcal{A}'$ the commutant of $\mathcal{A}$:

$$\mathcal{A}' = \{B \in \mathcal{B}(\mathcal{H}) : BA = AB \text{ for any } A \in \mathcal{A}\}.$$

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For any subset $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, $\mathcal{A}'$ is an algebra which contains the identity operator I in $\mathcal{B}(\mathcal{H})$; moreover, it is easy to check that $\mathcal{A}'$ is closed in the strong operator topology (equivalently, it is closed in the weak operator topology). If $\mathcal{A}$ is self-adjoint, then $\mathcal{A}'$ is a von Neumann algebra. In particular, if $\mathcal{C}$ is a von Neumann algebra, then $\mathcal{C}'$ is a von Neumann algebra (19). Let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $\mathcal{C}' \subset \mathcal{B}(\mathcal{H})$ its commutant. Then $\mathcal{C} \cap \mathcal{C}'$ is the common center of the algebras $\mathcal{C}$ and $\mathcal{C}'$. It is obvious that $\mathcal{C} \cap \mathcal{C}' \subset \mathcal{B}(\mathcal{H})$ is a (commutative) von Neumann algebra. A von Neumann algebra is called a factor if its center is equal to the set of all scalar multiple of the identity operator. Let $\mathcal{H}$ be a complex Hilbert space. A linear manifold in $\mathcal{H}$ is a subset of $\mathcal{H}$ which is closed under vector addition and under mutiplication by complex numbers. A subspace of $\mathcal{H}$ is a closed manifold. We shall often disregard the distinction between an orthogonal projection and its range space. Thus we consider a subspace lattice as consisting of orthogonal projections or subspaces and we may use the same notation to indicate either. This occurs most often in the technical arguments. Let $\mathcal{L}$ be a subset of all orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then $\mathcal{L}$ is called a lattice if $\mathcal{L}$ is closed under the operators "$\wedge$" and "$\vee$" for finitely many elements of $\mathcal{L}$. If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathcal{H}$, Alg$\mathcal{L}$ denotes the algebra of all bounded operators acting on $\mathcal{H}$ that leave invariant every orthogonal projection in $\mathcal{L}$, that is,

$$\text{Alg}\mathcal{L} = \{A \in \mathcal{B}(\mathcal{H}) : AE = EAE \text{ for any } E \in \mathcal{L}\}.$$

A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on a Hilbert space $\mathcal{H}$, containing $\mathbf{0}$ and I ($\mathbf{0}$ represents zero operator acting on $\mathcal{H}$). Dually, if $\mathcal{C}$ is a subalgebra of the set of all bounded operators acting on $\mathcal{H}$, then Lat$\mathcal{C}$ is the lattice of all orthogonal projections invariant for each operator in $\mathcal{C}$. An algebra $\mathcal{C}$ is reflexive if $\mathcal{C}$ = AlgLat$\mathcal{C}$. A lattice $\mathcal{L}$ is reflexive if $\mathcal{L}$ = LatAlg$\mathcal{L}$. A lattice $\mathcal{L}$ is commutative if each pair of orthogonal projections in $\mathcal{L}$ commutes. Especially, if $\mathcal{L}$ is a commutative subspace lattice, or CSL, then Alg$\mathcal{L}$ is called a CSL algebra. Subspace lattices need not be reflexive ; however, commutative ones are reflexive (4). If $f_1, f_2, \cdots, f_n$ are vectors in some Hilbert space, then $[f_1, f_2, \cdots, f_n]$ means the subspace generated by vectors $f_1, f_2, \cdots, f_n$.

2. Examples of Alg$\mathcal{L}$

EXAMPLE 1. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{ e_1, e_2, \cdots \}$ and let $\mathcal{L}$ be the lattice generated by $\{ [e_1, e_2], [e_3, e_4], [e_5, e_6], \cdots \}$. Then $\vee\mathcal{F} = \mathcal{H}$ and Alg$\mathcal{L}$ consists of matrices of the following form:

$$\begin{pmatrix} * & * & & & \\ * & * & & & \\ & & * & * & \\ & & * & * & \\ & & & & \ddots \end{pmatrix}$$

with respect to the basis $\{ e_1, e_2, \cdots \}$, where all non-starred entries are zeros. Since Alg$\mathcal{L}$ is self-adjoint, Alg$\mathcal{L}$ is a von Neumann algebra.

EXAMPLE 2. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{ e_1, e_2, \cdots \}$ and let $\mathcal{L}$ be the lattice generated by $\mathcal{F} = \{ [e_1, e_2], [e_3, e_4, e_5], [e_6, e_7, e_8, e_9], \cdots \}$. Since Alg$\mathcal{L}$ consists of matrices of the following form:

$$\begin{pmatrix} * & * & & & & & & & & \\ * & * & & & & & & & & \\ & & * & * & * & & & & & \\ & & * & * & * & & & & & \\ & & * & * & * & & & & & \\ & & & & & * & * & * & * & \\ & & & & & * & * & * & * & \\ & & & & & * & * & * & * & \\ & & & & & * & * & * & * & \\ & & & & & & & & & \ddots \end{pmatrix}$$

with respect to the basis $\{ e_1, e_2, \cdots \}$, where all non-starred entries are zeros, Alg$\mathcal{L}$ is a von Neumann algebra and Alg$\mathcal{L}$ = Alg$\mathcal{F}$.

EXAMPLE 3. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{ e_i : i = 1, 2, \cdots \}$ and let $\mathcal{L}_\infty$ be the subspace lattice generated by $\mathcal{F} = \{ [e_{2i-1}], [e_{2i-1}, e_{2i}, e_{2i+1}] : i = 1, 2, \cdots \}$. Then

$\vee\mathcal{F} = \mathcal{H}$ and Alg$\mathcal{L}_\infty$ consists of matrices of the following form :

$$\begin{pmatrix} * & * & & & & \\ & * & & & & \\ & * & * & * & & \\ & & & * & & \\ & & & * & \cdot & \\ & & & & & \ddots \end{pmatrix}.$$

with respect to the basis $\{ e_1, e_2, \cdots \}$, where all non-starred entries are zeros. Because Alg$\mathcal{L}_\infty$ is not self-adjoint, Alg$\mathcal{L}_\infty$ is not a von Neumann algebra. We know that Alg$\mathcal{L}_\infty$ is a tridiagonal algebra.

EXAMPLE 4. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{ e_1, e_2, \cdots \}$ and let $\mathcal{F} = \{ [e_i] : i = 1, 2, \cdots \}$ and let $\mathcal{L}$ be the lattice generated by $\mathcal{F}$. If A is in Alg$\mathcal{L}$, then A is the matrix which has the form :

$$\begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & \ddots \end{pmatrix}$$

with respect to the basis $\{ e_1, e_2, \cdots \}$, where all non-starred entries are zeros. Hence Alg$\mathcal{L}$ is a von Neumann algebra. Let $\{E_i\}$ be a subset of $\mathcal{L}$, where E_i is the orthogonal projection from $\mathcal{H}$ onto $[e_1, e_2, \cdots, e_i]$. Then $\{ E_i \}$ converges strongly to I. Since I is not in $\mathcal{L}$ and E_i is in $\mathcal{L}$ $(i = 1, 2, \cdots)$, $\mathcal{L}$ is not strongly closed. In particular, $\mathcal{L}$ is not complete.

3. General Theorems

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$. $\mathcal{C}$ is a von Neumann algebra if $\mathcal{C}$ is a C^*-algebra which is closed in the weak operator topology. If $\mathcal{L}$ is a family of orthogonal projections acting on $\mathcal{H}$, then Alg$\mathcal{L}$ is an algebra containing I and closed in the weak operator topology. Therefore in order to prove that Alg$\mathcal{L}$ is a von Neumann algebra, it is sufficient to show that Alg$\mathcal{L}$ is self-adjoint.

THEOREM 1. *Let $\mathcal{L}$ be a family of orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then*

(1) *Alg$\mathcal{L}$ is an algebra containing I (I represents the identity operator acting on $\mathcal{H}$) .*
(2) *Alg$\mathcal{L}$ is closed in the norm topology .*
(3) *Alg$\mathcal{L}$ is closed in the weak operator topology.*

LEMMA 2. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be families of orthogonal projections acting on a Hilbert space $\mathcal{H}$. If $\mathcal{L}_1 \subset \mathcal{L}_2$, then Alg$\mathcal{L}_2 \subset$ Alg$\mathcal{L}_1$.*

Proof. Let A be in Alg$\mathcal{L}_2$. Then $AE = EAE$ for all E in $\mathcal{L}_2$. Since $\mathcal{L}_1 \subset \mathcal{L}_2$, $AE = EAE$ for all E in $\mathcal{L}_1$. Hence A is in Alg$\mathcal{L}_1$.

Let E and F be orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then a partial order relation $\leq$ is described as follows : $E \leq F$ if and only if $EF = FE = E$. E, F are said to be mutually orthogonal if $EF = 0$. And the image $R(E) = \{Ef : f \in \mathcal{H}\}$ is called the rang space of E.

LEMMA 3. *Let $\mathcal{F}$ be a family of mutually orthogonal projections acting on $\mathcal{H}$. If $\mathcal{L}$ is the lattice generated by $\mathcal{F}$, then Alg$\mathcal{L}$ = Alg$\mathcal{F}$.*

THEOREM 4. *Let $\mathcal{H}$ be a separable Hilbert space let $\mathcal{F}$ be a family of mutually orthogonal projections acting on $\mathcal{H}$ such that $\vee\mathcal{F} = I$. If $\mathcal{L}$ is the lattice generated by $\mathcal{F}$, then Alg$\mathcal{L}$ is a von Neumann algebra.*

Proof. From Theorem 1, Alg$\mathcal{L}$ is algebra containing I and closed in the weak operator topology. Therefore it is sufficient to show that Alg$\mathcal{L}$ is self-adjoint. Let A be an element in Alg$\mathcal{L}$. Suppose that $\mathcal{F} = \{ E_1, E_2, \cdots \}$, where E_i is an orthogonal projection acting on $\mathcal{H}$ for all $i = 1, 2, \cdots$. Since A is in Alg$\mathcal{L}$, $AE_i = E_iAE_i$ for all $i = 1, 2, \cdots$. Since $AE_i^{\perp} = E_i^{\perp}AE_i^{\perp}$ for all $i = 1, 2, \cdots$, A^* is in Alg$\mathcal{L}$ by Lemma 3, i.e. Alg$\mathcal{L}$ is self-adjoint.

THEOREM 5. *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{F}$ be a mutually orthogonal family of subspaces of $\mathcal{H}$ and $\mathcal{L}$ be the lattice generated by $\mathcal{F}$. If $\vee\mathcal{F} \neq \mathcal{H}$, Alg$\mathcal{L}$ is not a von Neumann algebra.*

If E and F are orthogonal projections from a Hilbert space $\mathcal{H}$ onto closed subspaces Y and Z, respectively, then

$$E \leq F \text{ if and only if } Y \subset Z.$$

THEOREM 6. *Let $\mathcal{L}$ be a lattice of orthogonal projections acting on a separable Hilbert space $\mathcal{H}$ and let $\mathcal{F} = \{ F : F$ is a nonzero minimal element in $\mathcal{L}\}$. Then $\mathcal{F}$ is a mutually orthogonal family.*

Proof. Let E and F be elements of $\mathcal{F}$. Suppose that $E \wedge F \neq \mathbf{0}$ and $E \neq F$. Since $E \wedge F \leq F$ and F is minimal, $E \wedge F = F$. Hence $F \leq E$. Since E and F are minimal, $E = F$. So $E \wedge F = \mathbf{0}$ or $E = F$. Hence $\mathcal{F}$ is a mutually orthogonal family.

THEOREM 7. *Let $\mathcal{L}$ be a family of orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then Alg$\mathcal{L}$ is a von Neumann algebra if and only if Alg$\mathcal{L} = \mathcal{L}'$.*

Proof. Necessity : If A is in $\mathcal{L}'$, then $AE = EA$ for all E in $\mathcal{L}$. Since $AE = AEE = EAE$ for all E in $\mathcal{L}$, A is in Alg$\mathcal{L}$. Since Alg$\mathcal{L}$ is a von Neumann algebra, A^* is in Alg$\mathcal{L}$ for all A in Alg$\mathcal{L}$. If A is in Alg$\mathcal{L}$, then $AE = EAE$ and $A^*E = EA^*E$ for all E in $\mathcal{L}$. Hence $AE = EA$ for all E in $\mathcal{L}$. Thus A is in $\mathcal{L}'$.

Sufficiency : It is sufficient to show that Alg$\mathcal{L}$ is self- adjoint. Suppose that A is in Alg$\mathcal{L}$. Since Alg$\mathcal{L} = \mathcal{L}'$, $AE = EAE$ and $AE = EA$ for all E in $\mathcal{L}$. Hence for all E in $\mathcal{L}$ $EA = EAE$, that is, $A^*E = EA^*E$ for all E in $\mathcal{L}$. Therefore A^* is in Alg$\mathcal{L}$.

LEMMA 8(2). *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice of orthogonal projections acting on $\mathcal{H}$. Then $\mathcal{L}$ is reflexive.*

LEMMA 9. *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{L}$ be a complete lattice of orthogonal projections acting on $\mathcal{H}$. Let $\mathcal{F} = \{ F : F$ is a nonzero minimal element in $\mathcal{L}\}$. If E is a nonzero element in $\mathcal{L}$, then there exists E_0 in $\mathcal{F}$ such that $E_0 \leq E$.*

THEOREM 10. *Let $\mathcal{L}$ be a commutative subspace lattice of orthogonal projections acting on a separable Hilbert space $\mathcal{H}$. If Alg$\mathcal{L}$ is a von Neumann algebra, then there exists a family $\mathcal{F}$ of mutually orthogonal projections in $\mathcal{L}$ which generates completely $\mathcal{L}$.*

Proof. Let $\mathcal{F} = \{F : F$ is a nonzero minimal element in $\mathcal{L}\}$. Then $\mathcal{F}$ is a mutually orthogonal family by Theorem 6. We shall show that $\mathcal{L} = G(\mathcal{F})$, where $G(\mathcal{F})$ is the copmpete lattice generated by $\mathcal{F}$. Let E

be a nonzero element in $\mathcal{L}$. Suppose that E is not in $G(\mathcal{F})$. If $EF = 0$ for all F in $\mathcal{F}$, then $E(\vee\mathcal{F}) = \mathbf{0}$. Since E is in $\mathcal{L}$, there exists an element E_0 in $\mathcal{F}$ such that $E_0 \leq E$ by Lemma 9. Since $E_0(\vee\mathcal{F}) \neq \mathbf{0}$, $E(\vee\mathcal{F}) \neq \mathbf{0}$. It is a contradiction.

Suppose that $EF \neq 0$ for some $F \in \mathcal{F}$. Since $E \wedge F$ is in $\mathcal{L}$ and F is minimal, $E \wedge F = F$. Hence $F \leq E$. Put $\mathcal{F}_1 = \{D \in \mathcal{F} : D \leq E\}$. Then $\vee\mathcal{F}_1 \leq E$. Suppose that $\vee\mathcal{F}_1 = E$. Since $G(\mathcal{F})$ is complete, $\vee\mathcal{F}_1$ is in $G(\mathcal{F})$. It is a contradiction.

If $\vee\mathcal{F}_1$ is a proper subprojection of E, then $E - \vee\mathcal{F}_1$ is in $\mathcal{L}$. For each A in $\mathrm{Alg}\mathcal{L}$, by Theorem 7

$$\begin{aligned} A(E - \vee\mathcal{F}_1) &= AE - A(\vee\mathcal{F}_1) \\ &= EA - (\vee\mathcal{F}_1)A \\ &= (E - \vee\mathcal{F}_1)A. \end{aligned}$$

Hence $E-\vee\mathcal{F}_1$ is in $\mathrm{LatAlg}\mathcal{L}$. Since $\mathcal{L} = \mathrm{LatAlg}\mathcal{L}$ by Lemma 8, $E-\vee\mathcal{F}_1$ is in $\mathcal{L}$. By Lemma 9, there exists a nonzero minimal element E_1 in $\mathcal{L}$ such that $E_1 \leq E - \vee\mathcal{F}_1$. It is a contradiction. Since $\mathcal{L}$ contains $G(\mathcal{F})$, $\mathcal{L} = G(\mathcal{F})$.

THEOREM 11(19). *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a $*$-algebra of operators with I in $\mathcal{C}$. Then the following statements are equivalent.*

(1) *$\mathcal{C}$ is a von Neumann algebra.*
(2) *$\mathcal{C} = \mathcal{C}''$, where $\mathcal{C}''$ is the bicommutant of $\mathcal{C}$.*

Let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and let E be an orthogonal projection acting on a Hilbert space $\mathcal{H}$. We shall write

$$E\mathcal{C} = \{EA : A \in \mathcal{C}\}.$$

DEFINITION 12. Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and let $\mathcal{P}_{\mathcal{C}}$ be the set of orthogonal projections in $\mathcal{C}$.

(1) Two orthogonal projections E, F in $\mathcal{P}_{\mathcal{C}}$ are said to be equivalent, and this relation is denoted by $E \sim F$, if there exists a partial isometry U in $\mathcal{C}$ such that $E = U^*U$ and $F = UU^*$

; then $UE = U = FU$. We say that E is dominated by F, and we denote by $E \prec F$ this relation, if E is equivalent to a subprojection of F.

(2) An orthogonal projection E in $\mathcal{P}_\mathcal{C}$ is said to be abelian if $E\mathcal{C}E$ is commutative.

(3) An orthogonal projection E in $\mathcal{P}_\mathcal{C}$ is said to be finite if whenever $E \sim F \leq E$ for an orthogonal projection F in $\mathcal{P}_\mathcal{C}$, it follows that $F = E$.

(4) An orthogonal projection E in $\mathcal{P}_\mathcal{C}$ is said to be a central projection if it belongs to the center $\mathcal{C} \cap \mathcal{C}'$ of $\mathcal{C}$.

(5) An orthogonal projection E in $\mathcal{P}_\mathcal{C}$ is said to be properly infinite if whenever PE is finite, for each central projection P in $\mathcal{P}_\mathcal{C}$, it follows that $PE = 0$.

(6) The central cover $\mathcal{C}_A$ of A in $\mathcal{C}$ is the greatest lower bound of all central projection G in $\mathcal{P}_\mathcal{C}$ such that $GA = A$.

(7) An orthogonal projection E in $\mathcal{P}_\mathcal{C}$ is said to be faithful if $\mathcal{C}_E = I$.

Definition 13. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra.

(1) $\mathcal{C}$ is said to be finite if I is a finite orthogonal projection.

(2) $\mathcal{C}$ is said to be semifinite if any nonzero central projection contains a nonzero finite orthogonal projection.

(3) $\mathcal{C}$ is said to be of type I if any nonzero central projection contains a nonzero abelian orthogonal projection.

(4) $\mathcal{C}$ is said to be of type II if it is semifinite and it does not contain any nonzero abelian orthogonal projection.

(5) $\mathcal{C}$ is said to be of type III if it does not contain any nonzero finite orthogonal projection.

(6) $\mathcal{C}$ is said to be of type I_{fin} if it is finite and of type I.

(7) $\mathcal{C}$ is said to be of type I_∞ if it is not finite and it is of type I.

(8) $\mathcal{C}$ is said to be of type II_1 if it is finite and of type II.

(9) $\mathcal{C}$ is said to be of type II_∞ if it is not finite, but it is of type II.

Theorem 14(19). *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. $\mathcal{C}$ is of type I if and only if any nonzero orthogonal projection in $\mathcal{C}$ contains an abelian nonzero orthogonal projection.*

LEMMA 15. *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then $\mathcal{C}$ is finite if and only if $\dim\mathcal{H} < \infty$.*

LEMMA 16. *Let $\mathcal{H}$ be a separable infinite Hilbert space with an orthonormal basis $\{ e_1, e_2, \cdots \}$ and let $\mathcal{F} = \{ [e_i] : i = 1, 2, \cdots \}$. If $\mathcal{L}$ is th lattice generated by $\mathcal{F}$, then $Alg\mathcal{L}$ is of type I_∞.*

THEOREM 17. *Let $\mathcal{H}$ be a separable infinite Hilbert space and let $\mathcal{F}$ be a family of mutually orthogonal projections acting on $\mathcal{H}$ such that $\vee\mathcal{F} = I$. If $\mathcal{L}$ is the lattice generated by $\mathcal{F}$, then $Alg\mathcal{L}$ is of type I_∞.*

Proof. Suppose that $\mathcal{F} = \{ E_1, E_2, \cdots \}$ and $\mathcal{H}_i$ is the subspace of $\mathcal{H}$ such that $E_i(\mathcal{H}) = \mathcal{H}_i$ for all $i = 1, 2, \cdots$. Let A be in Alg$\mathcal{L}$. Since Alg$\mathcal{L}$ = Alg$\mathcal{F}$ by Lemma 3, A is in Alg$\mathcal{F}$. Hence A has the following matrix form on $\sum_i \oplus\mathcal{H}_i$

$$\begin{array}{c}
\begin{array}{cccc} \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 & \cdots \end{array} \\
\begin{pmatrix} A_{11} & & & \\ & A_{22} & & 0 \\ & & A_{33} & \\ & 0 & & \ddots \end{pmatrix}
\end{array}$$

where $A_{ii} : \mathcal{H}_i \to \mathcal{H}_i$ is the operator such that $A_{ii} = A\mid_{\mathcal{H}_i}$ for all $i = 1, 2, \cdots$.

Let E be a nonzero orthogonal projection in Alg$\mathcal{L}$. Then E has the following matrix form on $\sum_i \oplus\mathcal{H}_i$:

$$\begin{pmatrix} E_{11} & 0 & 0 & 0 & \cdots \\ 0 & E_{22} & 0 & 0 & \cdots \\ 0 & 0 & E_{33} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \\ & & & & \end{pmatrix}$$

where E_{ii} is the orthogonal projection acting on $\mathcal{H}_i$ such that $E_{ii} = E\mid_{\mathcal{H}_i}$ for all $i = 1, 2, \cdots$ and E_{kk} is nonzero for some k. If E_{kk} is a nonzero orthogonal projection acting on $\mathcal{H}_k$ for some k, E_{kk} contains a subprojection F_{kk} of rank one. Let F be the orthogonal projection acting on $\sum_i \oplus\mathcal{H}_i$ such that $E_kF\mid_{\mathcal{H}_k} = F_{kk}$ and $E_iF\mid_{\mathcal{H}_j} = 0$ if $i \neq k$ or $j \neq k$. Then F is in Alg$\mathcal{L}$ and F is a nonzero abelian subprojection of E. Hence Alg$\mathcal{L}$ is of type I by Theorem 14. By Lemma 15, Alg$\mathcal{L}$ is of type I_∞.

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be any set of bounded operators and E be an orthogonal projection acting on $\mathcal{H}$. We shall write

$$\mathcal{M}_E = \{ES\,|_{R(E)} : S \in \mathcal{M}\} \subset \mathcal{B}(R(E)).$$

THEOREM 18(19). *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and E in $\mathcal{P}_{\mathcal{C}}$. Then*

(1) $\mathcal{C}_E \subset \mathcal{B}(R(E))$ *and* $(\mathcal{C}')_E \subset \mathcal{B}(R(E))$ *are von Neumann algebras.*
(2) $(\mathcal{C}_E)' = (\mathcal{C}')_E$.
(3) $E\mathcal{C}E \cong \mathcal{C}_E$ *($*$-isomorphism of $*$-algebras).*

PROPOSITION 19(20). *Let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then $\mathcal{C}$ is of type I if and only if $\mathcal{C}$ contains a faithful abelian orthogonal projection.*

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be any set of bounded operators and let $\mathcal{X} \subset \mathcal{H}$ be any set of vectors. For notational convenience we define

$$\mathcal{M}\mathcal{X} = \{Af : A \in \mathcal{M}, f \in \mathcal{X}\}.$$

We say that $\mathcal{X}$ is cyclic for $\mathcal{M}$ if $[\mathcal{M}\mathcal{X}] = \mathcal{H}$. We call $\mathcal{X}$ separating for $\mathcal{M}$ if $A \in \mathcal{M}$ and $Af = 0$ for all $f \in \mathcal{X}$ imply $A = 0$. If $\mathcal{X} = \{ f \}$ consists of one vector f, we apply these terms to the vector itself.

PROPOSITION 20. *Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a $*$-algebra of bounded operators. Then*

(1) *A vector f in $\mathcal{H}$ is cyclic for $\mathcal{C}$ if and only if f is separating for $\mathcal{C}'$.*
(2) *For any vector f in $\mathcal{H}$, the orthogonal projection E from $\mathcal{H}$ onto $[\mathcal{C}f]$ belongs to $\mathcal{C}'$.*

Proof. (1) Let f be cyclic and suppose that $A' \in \mathcal{C}'$ with $A'f = 0$. Then $A'Af = AA'f = 0$, for each A in $\mathcal{C}$, and hence A' vanishes on $[\mathcal{C}f] = \mathcal{H}$, so $A' = 0$.

Conversely, if E is the orthogonal projection from $\mathcal{H}$ onto $[\mathcal{C}f]$, then $Ef = f$, since I is in $\mathcal{C}$. Now if A and B are elements in $\mathcal{C}$, then ABf is in $\mathcal{C}f$, so that $\mathcal{C}f$ is invariant under $\mathcal{C}$, as is, therefore $[\mathcal{C}f]$. This means that $EAE = AE$ for each A in $\mathcal{C}$, so that also $EA^*E = A^*E$, because $\mathcal{C}$ is $*$-algebra. Thus $EA = EAE = AE$ on taking adjoints, and E is in $\mathcal{C}'$. Now $(I - E)f = f - f = 0$, and because $I - E$ is in $\mathcal{C}$ and f is separating for $\mathcal{C}'$, $I = E$ or $[\mathcal{C}f] = \mathcal{H}$. So f is cyclic for $\mathcal{C}$. This proves (1) and incidentally (2).

THEOREM 21(20).. *If an abelian von Neumann algebra $\mathcal{C}$ has a cyclic vector, then $\mathcal{C}$ is maximal abelian, i.e., $\mathcal{C} = \mathcal{C}'$.*

LEMMA 22(20).. *Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If E in $\mathcal{C}$ is a faithful orthogonal projection, then $\mathcal{C}' \cong \mathcal{C}'E$.*

THEOREM 23. *Let $\mathcal{H}$ be a separable Hilbert space. If $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ is a of type I von Neumann algebra, then $\mathcal{C}'$ is also of type I.*

Proof. First suppose $\mathcal{C}$ is abelian and let G be a nonzero central projection. Take any nonzero vector f in $R(G)$ and let E be the orthogonal projection from $\mathcal{H}$ onto $[\mathcal{C}f]$. By Proposition 20(2), E is in $\mathcal{C}'$. Also $E \leq G$ and by Theorem 11 and Theorem 18, $(\mathcal{C}')_E = (\mathcal{C}_E)'$. But $\mathcal{C} \subset \mathcal{C}'$, so $\mathcal{C}_E \subset (\mathcal{C}')_E = (\mathcal{C}_E)'$ and since f is cyclic for $\mathcal{C}_E$ on $R(E)$, Theorem 21 gives $\mathcal{C}_E = (\mathcal{C}_E)'$. Thus $E\mathcal{C}'E \cong (\mathcal{C}')_E = (\mathcal{C}_E)' = \mathcal{C}_E$ by Theorem 18, so E is an abelian orthogonal projection for $\mathcal{C}'$. For the general case, let F be a faithful abelian orthogonal projection in $\mathcal{C}$ by Proposition 19. Then $\mathcal{C}' \cong \mathcal{C}'F \cong (\mathcal{C}')_F = (\mathcal{C}_F)'$ by Theorem 18 and Lemma 22. Also $\mathcal{C}_F \cong F\mathcal{C}F$ (abelian), by Theorem 18. By the abelian case, $\mathcal{C}'$ is of type I.

From Theorem 17 and Theorem 23, we can get the following corollary.

COROLLARY 24. *Let $\mathcal{H}$ be a separable infinite Hilbert space and let $\mathcal{F}$ be a family of mutually orthogonal projections acting on $\mathcal{H}$ such that $\vee\mathcal{F} = I$. If $\mathcal{L}$ is the lattice generated by $\mathcal{F}$, then $(Alg\mathcal{L})'$ is of type I.*

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