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ON THE TIGHTNESS OF $C_p(X)$

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1. Introduction.

We assume that all spaces in this paper are Tychonoff spaces. Let $C_p(X)$ be the space of all real-valued continuous functions on X with the topology of pointwise convergence.

The topological space $C_p(X)$ can only be described in terms of convergent sequences. We note that the property of being sequential, and also the even more general property of being a k-space, are equivalent to the Fréchet-Urysohn property for spaces of type $C_p(X)$. This very subtle and unexpected result was obtained by E. G. Pytkeev (see [10]), J. Gerlits (see [4]) , and V. V. Uspenskii (see [11]). The principal significance of the irreducibility of the topology of $C_p(X)$ to convergent sequences and compact sets is the following fact: when X is compact, then the closure operator in $C_p(X)$ is completely determined by its behavior on a countable set (see [1], [9]). So the tightness plays an important role in the C_p-theory.

The aim of this paper is to study some useful cardinal invariants (Fréchet-Urysohn property, tightness, netweight) related to the properties of X and $C_p(X)$. We obtain some sufficient conditions for a space X that every compact (countably compact) subspace of $C_p(X)$ is a Fréchet-Urysohn space. And we show that for a functionally complete Lindelöf space, any closed bounded subspace has a countable tightness. Moreover, if X is a compact submetrizable space, then it is a Lindelöf Σ-space with a countable weight.

2. Fréchet-Urysohn spaces.

The *Lindelöf number* of a space X, denoted by $l(X)$, is the smallest infinite cardinal τ such that any open cover of X has a subcollection with cardinality $\leqslant \tau$ which covers X (see [2]).

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The tightness of a space X, denoted by $t(X)$, is the smallest infinite cardinal τ such that for any set $A \subset X$ and any point $x \in cl(A)$ there is a set $B \subset A$ for which $|B| \leq \tau$ and $x \in cl(B)$ (see [6]).

A space X is called a *Fréchet-Urysohn space* if for each set $A \subset X$ and each point $x \in cl(A)$ there is a sequence $\{x_n : n \in N\}$ in A converging to x (see [2]). From these definitions it is obvious that every Fréchet-Urysohn space has a countable tightness. But the converse is negative. For example, let I be the closed unit interval in the real line R, then $C_p(I)$ is not a Fréchet-Urysohn space but the tightness of $C_p(I)$ is countable since I is compact. In [5], M. Ismail and P. Nyikos proved that for a space X with a base of subinfinite rank, X is Fréchet-Urysohn if and only if X is of countable tightness.

The following theorem is a very important result obtained by A. V. Arhangelskii and E. G. Pytkeev.

THEOREM 2.1([1], [9]). *$l(X^n) \leq \tau$ for all $n \in N$ if and only if $t(C_p(X)) \leq \tau$.*

The following is immediate.

REMARK 2.2. *If X is a space with countable tightness, then the tightness of every subspace of X is also countable.*

A map is called *perfect* if it is continuous, closed, and preimage of points are compact. We call a space X a *Lindelõf Σ-space* if it can be represented as a continuous image of a space Y which can be perfectly mapped onto a second countable space (see [7]). We note that the class of Lindelöf Σ-spaces is very stable: it is closed under countably many products, transition to a continuous image, and transition to a closed subspace.

The least infinite cardinal of a basis for a space X is called the *weight* of X and denoted by $w(X)$. A *net* for a space X is a collection $\mathcal{N}$ of subsets of X such that every open set in X is the union of elements of $\mathcal{N}$. The *netweight* of X, denoted by $nw(X)$, is the least infinite cardinal of a net for a space X. A space X is said to be *τ-monolithic* if $nw(A) \leq \tau$ for all $A \subset X$ with $|A| \leq \tau$. We say that X is *monolithic* if it is τ-monolithic for all τ. According to A. V. Arhangelskii [2], a subspace of a monolithic space is also monolithic and if X is a Lindelöf Σ-space or a compact space, then $C_p(X)$ is monolithic.

THEOREM 2.3. *If X is a Lindelöf Σ-space, then every compact subspace of $C_p(X)$ is a Fréchet-Urysohn space and hence the tightness of X is countable.*

Proof. Let Y be a compact subspace of $C_p(X)$, $y \in Y$, $A \subset Y$ and $y \in cl(A)$. It is enough to show that there is a sequence in A converging to y. Since X is a Lindelöf Σ-space, X^n is also a Lindelöf Σ- space for all $n \in N$. So there is a continuous onto map $f : Z \longrightarrow X^n$, where Z can be mapped by a perfect map onto a second countable space. Since every second countable space is Lindelöf and a preimage of a Lindelöf space under a perfect map is Lindelöf, Z is a Lindelöf space. Thus X^n is Lindelöf for all $n \in N$. Hence the tightness of $C_p(X)$ is countable by Theorem 2.1. According to Remark 2.2, there is a countable set $B \subset A$ such that $y \in cl(B)$. From the fact that X is a Lindelöf Σ-space, it follows that $C_p(X)$ is monolithic. Since every subspace of a monolithic space is monolithic, $cl(B)$ is monolithic. Moreover, $cl(B)$ has a countable dense subspace B, i.e., $cl(B)$ is separable. So, $nw(cl(B)) = d(cl(B)) = \aleph_0$, where $d(X)$ is a density of X. Since Y is compact, $cl(B)$ is also compact. Cleary, the netweight and the weight of a compact space are precisely same, and so $nw(cl(B)) = w(cl(B))$, i.e., $cl(B)$ is second countable. Since $y \in cl(B)$, there is a sequence $\{y_n : n \in N\}$ in $B(\subset A)$ which converges to y, and therefore Y is a Fréchet-Urysohn space. $\square$

EXAMPLE 2.4. *Let I be the closed unit interval* $[0, 1]$. *Then I is a Lindelöf Σ-space. So every compact set K in $C_p(I)$ is a Fréchet-Urysohn space by Theorem 2.3. But, by the result in [6, Exercises 8(a)], $C_p(I)$ is not a Fréchet-Urysohn space.*

This example ensures that Theorem 2.1 cannot be extended to a Fréchet-Urysohn property. In other words, for a compact space X (thus X^n is Lindelöf for all $n \in N$), $C_p(X)$ need not be a Fréchet-Urysohn space.

For further study, we need the following concepts.

We call a set $F \subset C_p(X)$ *separates points in* X if for any distinct points x and y in X, there is a point $f \in F$ such that $f(x) \neq f(y)$. A space X is said to be *functionally complete* if there exists a compact set $F \subset C_p(X)$ which separates points in X. A functionally complete compact space is called an *Eberlein compact space* (see [2]).

THEOREM 2.5. *If X is an Eberlein compact space, then X is a Fréchet-Urysohn space.*

Proof. Let X be an Eberlein compact space. Then there is a compact set $F \subset C_p(X)$ which separates points in X. Then for each $x \in X$ there is a map $g_x : F \longrightarrow R$ (R is the real numbers set) defined by $g_x(f) = f(x)$ for all $f \in F$. Putting $\pi_F(x) = g_x$ for all $x \in X$ gives the canonical evaluation map $\pi_F : X \longrightarrow C_p(F)$. Since F separates points in X, π_F is injective. By the compactness of X, X is homeomorphic.to a subspace of $C_p(F)$. By Theorem 2.1, the tightness of $C_p(F)$ is countable. The rest of the proof runs as Theorem 2.3. $\square$

In [3], A. V. Arhangelskii gave a problem (Q1053) that if X is a Lindelöf space and Y is a compact subspace of $C_p(X)$, then is the tightness of Y countable? For this problem, we are interested in finding a sufficient condition for a space X that every countably compact subspace of $C_p(X)$ has the countable tightness.

For a k-separable space, we obtain a strong result.

THEOREM 2.6. *If X is a k-separable space (i.e., X has a σ-compact dense subspace), then every countably compact subspace of $C_p(X)$ is a Fréchet-Urysohn space.*

Proof. Let Y be a countably compact subspace of $C_p(X)$ and A be an arbitrary subspace in Y. Then $cl_Y(A)$ is countably compact. Let $f \in cl_Y(A) \subset cl_{C_p(X)}(A)$. Then there is an element $g \in C_p(X)$ and a sequence $\{g_n : n \in N\}$ in A such that $\{g_n(x) : n \in N\}$ converges to $g(x) = f(x)$ for all $x \in D$, where D is a σ-compact dense subspace of X (see [8,Theorem 2.1]). Since R is a Tychonoff space, $f = g$. Thus Y is a Fréchet-Urysohn space. $\square$

In particular, every separable space is k-separable, and therefore we have the following corollary.

COROLLARY 2.7. *If X is a separable space, then every countably compact subspace of $C_p(X)$ is a Fréchet-Urysohn space.*

3. Countable tightness.

THEOREM 3.1. *If X is a functionally complete Lindelöf space, then the tightness of every closed bounded subspace of X is countable.*

Proof. Let Y be a closed bounded subspace of X. Since X is functionally complete, there is a compact set $F \subset C_p(X)$ separating points of X. We define a map $\varphi : X \longrightarrow C_p(F)$ by $\varphi(x)(f) = f(x)$ for all $f \in F$. Then φ is continuous. Moreover, if $x \neq y$ in X, then there is an element $f \in F$ such that $f(x) \neq f(y)$. Thus $\varphi(x)(f) \neq \varphi(y)(f)$ and so φ is one-to-one. From the result in [2, Proposition 2.5], every closed bounded subspace Y of X is compact. So $\varphi(Y)$ is embedded into $C_p(F)$. By the compactness of F, the tightness of $C_p(F)$ is countable. Since the tightness of every subspace in a space with countable tightness is also countable, the tightness of $\varphi(Y)$ is countable. Hence the tightness of Y is countable. □

COROLLARY 3.2. *If X is a functionally complete σ-compact space, then the tightness of any closed bounded subspace in X is countable.*

We call a space X is ***submetrizable*** if it is mapped onto a metrizable space by a one-to-one continuous map (see [6]).

THEOREM 3.3. *If X is a compact submetrizable space, then X is a Lindelöf Σ-space with a countable weight (hence the tightness of X is countable).*

Proof. Let X be a compact submetrizable space. Then $C_p(X)$ is monolithic and separable. So, $C_p(C_p(X))$ is a Lindelöf Σ-space (see [6, Exercises 10(e)]). Since a compact space X can be embedded into $C_p(C_p(X))$, X is homeomorphic to a closed subspace of $C_p(C_p(X))$. By the fact that every closed subspace of a Lindelöf Σ-space is a Lindelöf Σ-space, it follows that X is also a Lindelöf Σ-space. Let D be a countable dense subspace of $C_p(X)$. By the monolithicity of $C_p(X)$, we have that $nw(C_p(X)) = nw(cl(D)) = |D| = \aleph_0$. Thus $C_p(X)$ has a countable netweight. Since $nw(X) = nw(C_p(X))$, X also has a countable netweight. It is well-known that the weight and the netweight of a compact space are precisely the same, so the weight of X is countable. □

The following corollary is an immediate consequence.

COROLLARY 3.4. *If X is a compact metrizable space, then X is a Lindelöf Σ-space with a countable weight.*

By combining Theorem 3.3 with Theorem 2.3, we have the following result.

COROLLARY 3.5. *If X is a compact submetrizable space, then every compact subspace of $C_p(X)$ is a Fréchet-Urysohn space.*

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