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NEAR-RINGS WITH CHAIN CONDITIONS AND NIL-DERIVATIONS

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1. Introduction

In this paper, most of all our near-rings N are zero-symmetric (right) near-rings with multiplicative center $Z(N)$. In section 2, we will introduce the concepts of $GDCC$ and $GACC$ for ideals of N which are more generalized concepts of DCC and ACC for ideals of N. Also we can consider the concepts of $GDCC$ and $GACC$ for N-subgroups of N. Several important properties of near-rings with $GDCC$(resp. $GACC$) for ideals are investigated and some of the well known properties of ring with DCC(resp. ACC) or properties of near-ring with DCC(resp. ACC) on ideals are generalized to the properties of near-ring with $GDCC$(resp. $GACC$) on ideals. However for S-unital near-ring, the notion of $GDCC$(resp. $GACC$) on ideals is equivalent to that of DCC(resp. ACC) on ideals. These statements are motivated from the H.Tominaga's paper [18] in 1980. C.Faith [12] and I.N.Herstein [14] studied for rings with ascending chain condition on principal annihilator left ideals and descending chain condition on principal left ideals.

We can study the chain conditions on principal N-subgroups and on principal annhilator left ideals of near-ring as ring case, and investigate relationships between K-regularity, generalized bipotence and on $GDCC$ principal N-subgroups for S-unital near-rings.

In section 3, we will study the derivation on near-rings, a derivation on N is defined to be an additive endomorphism D satisfying the product rule $D(ab) = D(a)b + aD(b)$ for all a, b in N. An element a of N for which $D(a) = 0$ is called constant. For a, b in N, the symbol $[a, b]$ will denote the commutator $ab - ba$, the symbol (a, b) will denote

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the additive commutator $a+b-a-b$ and the symbol $a \circ b$ will denote the skew commutator $ab+ba$.

A derivation D will be called *centralizing* if $[D(a),\ a] \in Z(N)$ for all a in N, *skew centralizing* if $D(a) \circ a \in Z(N)$ for all a in N, in particualr D is called *commuting* if $[D(a),\ a]=0$ for all a in N, *skew commuting* if $D(a) \circ a = 0$ for all a in N and an element a of N with $[D(a),\ a]=0$ is called *commuting.* In ring theory, lots of mathematicians , for instance, H.E. Bell and G. Mason [3] in 1987, M. Bresar [5], [6], [7] in 1993, [4] in 1995, studied these concepts on endomorphisms, automorphisms or derivations of prime rings which are derived commutativity.

We shall investigate several charaterizations of near-ring with detivation and derive that any prime near-ring with derivation and certain conditions becomes a commutative ring, in order to preperation for proving our theorem, we begin with several useful lemmas.

Finally we consider nilpotent and nil derivation on N. In ring theory these concepts are studied by L.O.Chung and J.Luh [9], [10], [11] in 1985, and P. Grzeszczuk [13] in 1992. we shall prove that for a left strongly prime near-ring, in particular, prime near-ring with DCC on left annihilators, every nil derivation D on a non-zero left ideal of N is also nil on N.

2. Near-Rings with Generalized Chain Conditions

Now we will introduce the concepts of generalized chain conditions for near-rings that is generalized descending chain condition and generalized ascending chain condition with respect to ideals and N-subgroups of near-rings. For an ideal I of N and an N-subgroup M of N, we denote that, $I^{i}M = \{x \in N | I^{i}x \subset M\}$, for all positive integer i, we have the following ascending chain

$$M \subset I^{-1}M \subset I^{-2}M \subset \cdots \subset I^{-i} \subset \cdots .$$

We say that N satisfies the ***generalized descending chain condition (abbr. GDCC)*** for ideals if for every descending chain $M_1 \supset M_2 \supset M_3 \supset \cdots$ of ideals of N, there exist positive integers p, q such that $N^{p}M_{q} \subset M_i$ for each positive integer i. Dually N satisfies the ***generalized ascending chain condition (abbr. GACC)*** for ideals if for every ascending chain of $M_1 \subset M_2 \subset M_3 \subset \cdots$ ideals of N, there exist positive integers p, q such that $M_i \subset N^{-p}M_q$ for each positive integer i.

First we obtain the following remarks of the case 1, if N has the DCC for ideals then N has the $GDCC$ for ideals, case 2, if N has the ACC for ideals then N has the $GACC$ for ideals, but not conversely in general for the case 1 and case 2.

Indeed, for the case 1, let $M_1 \supset M_2 \supset M_3 \supset \cdots$ be any descending chain of ideals of N. Since N has the DCC for ideals, there exists a positive integer q such that $M_q = M_{q+1} = \cdots$, that is, $M_q \subset M_i$ for all positive integer i, for each fixed positive integer p, we get

$$N^p M_q \subset NM_q \subset M_q \subset M_i$$

for all positive integer i. Consequently N has the $GDCC$ for ideals. For the case 2, let $M_1 \subset M_2 \subset M_3 \subset \cdots$ be arbitrary ascending chain of ideals. From the fact that N has the ACC for ideals, there exists a positive integer q such that $M_q = M_{q+1} = \cdots$, that is, $M_i \subset M_q$ for all positive integer i Thus we obtain for each fixed positive integer p,

$$N^p M_i \subset NM_i \subset M_i \subset M_q, \ \textit{that is}, \ M_i \subset N^{-p}M_q,$$

for all positive integer i. Hence N has the $GACC$ for ideals. Next there are several examples which are near-ring with $GDCC$ or $GACC$ for ideals but not DCC or ACC for ideals in the following.

(1) Any near-ring with trivial multiplication satisfies both the $GDCC$ and the $GACC$ for ideals.

(2) Every nilpotent near-ring has both the $GDCC$ and the $GACC$ for ideals, for concrete examples $N = \{\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} | a \in \mathbb{Z}\}$ is nilpotent near-ring which has the $GDCC$ and DCC for ideals, similarly for $N = \{\begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} | q \in \mathbb{Q}\}$.

(3) $N = \{\begin{pmatrix} q_1 & 0 \\ q_2 & 0 \end{pmatrix} | q_1, q_2 \in \mathbb{Q}\}$ is non-nilpotent near-ring, has the $GDCC$ but not the DCC for ideals.

(4) $N = \{\begin{pmatrix} a & 0 \\ q & 0 \end{pmatrix} | a \in \mathbb{Z}, q \in \mathbb{Q}\}$ is a near-ring with the $GACC$ but not the ACC for ideals.

(5) The p-adic group $\mathbb{Z}_{p^\infty}$ with trivial multiplication is a near-ring. Using the similar properties of ring ideals, we see that it has the $GDCC$ but not has the DCC for ideals.

PROPOSITION 2.1. *The following statements are equivalent*

(1) *N satisfies the $GDCC$ for ideals.*
(2) *For each descending chain $M_1 \supset M_2 \supset M_3 \supset \cdots$ of ideals of N, there exists a positive integer p such that $N^p M_p \subset M_i$ for all positive integer i.*
(3) *Every direct summand ideal of N satisfies the $GDCC$ for ideals of N.*
(4) *For each non-empty family $\mathcal{F}$ of ideals of N, there exists an ideal K in $\mathcal{F}$ and a positive integer p such that $N^p K \subset J$ for all J in $\mathcal{F}$ satisfying $J \subset K$.*

Proof. (2) $\Longrightarrow$ (1). It is obvious from the definition of $GDCC$ for ideals. To show that (1) $\Longrightarrow$ (2), assume that N satisfies the $GDCC$ for ideals. Let $M_1 \supset M_2 \supset M_3 \supset \cdots$ be a descending chain of ideals of N. Since N has the $GDCC$ for ideals, there exist positive integers r and q, we have $N^r K_q \subset K_i$ for all positive integer i. Putting $p = \max\{r, q\}$ we see that $N^p K_p \subset N^r K_q \subset K_i$ for all positive integer i.
(1) $\Longleftrightarrow$ (3). It is clear.

(4) $\Longrightarrow$ (1). The condition (4) implies that N satisfies $GDCC$ for ideals by using the notion of $GDCC$ with respect to any descending chain of ideals of N. Now we will show that (1) $\Longrightarrow$ (4). Suppose that N has the $GDCC$ for ideals, and $\mathcal{F}$ is a non-empty family of ideals of N. Assume the condition (4) does not hold for $\mathcal{F}$. Take any $K_1 \in \mathcal{F}$. Then there exists an element $K_2 \in \mathcal{F}$ such that $K_1 \supset K_2$ but $NK_1 \not\subset K_2$. For this K_2 there exists an element K_3 in $\mathcal{F}$ such that $K_2 \supset K_3$ and $N^2K_2 \not\subset K_3$. Continuing these procedure till $n-2$ step, for K_{n-1} in $\mathcal{F}$, there exists an element K_n in $\mathcal{F}$ we have that $K_{n-1} \supset K_n$ and $N^{n-1}K_{n-1} \not\subset K_n$ and so forth. Thus there exists a descending chain $M_1 \supset M_2 \supset M_3 \supset \cdots$ of ideals such that $N^{n-1}K_{n-1} \not\subset K_n$ for all positive integer n. This statement violates the hypothesis of N which satisfies $GDCC$ for ideals. □

We obtain the dual statements of the previous proposition as following statement.

PROPOSITION 2.2. *The following statements are equivalent.*

(1) *N satisfies the $GACC$ for ideals.*
(2) *For every ascending chain $M_1 \subset M_2 \subset M_3 \subset \cdots$ of ideals of N, there exists a positive integer p such that $M_i \subset N^{-p}M_p$ for all positive integer i.*
(3) *Every ideal of N satisfies the $GACC$ for ideals of N.*
(4) *For each non-empty family $\mathcal{F}$ of ideals of N, there exists an element K of $\mathcal{F}$ and a positive integer p such that $J \subset N^{-p}K$ for all J in $\mathcal{F}$ satisfying $K \subset J$.*

LEMMA 2.3. *Let and N_1 be N_2 two near-rings, and let $f : N_1 \to N_2$ be a near-ring epimorphism. Then*

(1) *If N_1 satisfies the $GDCC$ on ideals, then so is N_2.*
(2) *If N_1 satisfies the $GACC$ on ideals, then so is N_2.*

Proof. (1) Let $K_1 \supset K_2 \supset K_3 \supset \cdots$ be a descending chain of ideals of N_2, then we have that

$$f^{-1}(K_1) \supset f^{-1}(K_2) \supset f^{-1}(K_3) \supset \cdots$$

is a descending chain of ideals of N_1. By hypothesis of N_1 which satisfies the $GDCC$ on ideals, there exists a positive integer p such that $N_1^p f^{-1}(K_p) \subset f^{-1}(K_i)$ for all positive integer i. Using the proposition 2.1 (2). This implies that $\{f^{-1}(N_2)\}^p \subset f^{-1}(K_i)$, that is, $f^{-1}(N_2^p K_p) \subset f^{-1}(K_i)$ for all positive integer i. Since f is an epimorphism we see that $N_2{}^p K_p \subset K_i$ for all positive integer i. Consequently N_2 satisfies the $GDCC$ on ideals.
(2) Assume that N_1 satisfies the $GACC$ on ideals. Let $K_1 \subset K_2 \subset K_3 \subset \cdots$ be an ascending chain of ideals of N_2. Then we obtain that

$$f^{-1}(K_1) \subset f^{-1}(K_2) \subset f^{-1}(K_3) \subset \cdots$$

is an ascending chain of ideals of N_1. Because N_1 satisfies the $GACC$ on ideals, there exists a positive integer p such that $N_1{}^p f^{-1}(K_i) \subset f^{-1}(K_p)$ for all positive integer i, from the proposition 2.2 (2). This inclusion implies that $f^{-1}(N_2{}^p K_i) \subset f^{-1}(K_p)$ for all positive integer i. Since f is an epimorphism we know that $N_2{}^p K_i \subset K_p$, that is, $K_i \subset N_2{}^{-p} K_p$ for all positive integer i. Hence N_2 has the $GACC$ on ideals. □

The following important statements are generalization of theorem 2.35 in [16] for near-rings.

THEOREM 2.4. *Let I be a direct summand ideal of any left near-ring N. Then*

(1) *N satisfies the $GDCC$ on ideals if and only if both I and N/I satisfy the $GDCC$ on ideals.*
(2) *N satisfies the $GACC$ for ideals if and only if both I and N/I satisfy the $GACC$ for ideals.*

Proof. The only if parts of (1) and (2) follow from the propositions 2.1 (3), 2.2 (3) and Lemma 2.3 (1) and (2). It is sufficient to prove the if parts of the statements (1) and (2).
(1) Assume that both I and N/I satisfy the $GDCC$ on ideals. Let $K_1 \supset K_2 \supset K_3 \supset \cdots$ be a descending chain of ideals of N. Then

$$K_1 \cap I \supset K_2 \cap I \supset K_3 \cap I \supset \cdots$$

is a descending chain of ideals of I and

$$(K_1 + I)/I \supset (K_2 + I)/I \supset (K_3 + I)/I \supset \cdots$$

is a descending chain of ideals of N/I. Applying the proposition 2.1 (2), there exists a positive integer p such that

$$N^p\{(K_p + I)/I\} \subset (K_\imath + I)/I$$

and

$$N^p(K_p \cap I) \subset K_\imath \cap I$$

for all positive integer I. Taking any positive integer j and any $r_1, r_2, \ldots, r_p$ in N, for each element x in K_p we obtain

$$r_1 r_2 \cdots r_p(x + I) \subset N^p(K_p + I) \subset K_{p+j} + I.$$

Since $0 \in I$, there exist k in K_{p+j} and a in I, $r_1 r_2 \cdots r_p x = k + a$. But, we see that $a \in K_p$ so that $a \in K_p \cap I$. Hence

$$N^p a \subset N^p(K_p \cap I) \subset K_{p+j} \cap I \subset K_{p+j}.$$

Finally,

$$N^p r_1 r_2 \cdots r_p x = N^p(k+a) \subset N^p k + N^p a \subset K_{p+j}.$$

Thus $N^{2p}x \subset K_{p+j}$, that is, $N^{2p}K_p \subset K_{p+j}$. Since j is arbitrary positive integer, $N^{2p}K_p \subset K_i$ for each positive integer $i > p$. By proposition 2.1, N satisfies the $GDCC$ on ideals.

(2) Assume that I and N/I have both the $GACC$ on ideals. Let $K_1 \subset K_2 \subset K_3 \subset \cdots$ be an ascending chain of ideals of N_2. Then we get that

$$K_1 \cap I \subset K_2 \cap I \subset K_3 \cap I \subset \cdots$$

is an ascending chain of ideals of I and

$$(K_1 + I)/I \subset (K_2 + I)/I \subset (K_3 + I)/I \subset \cdots$$

is an ascending chain of ideals of N/I. So that, there exists a positive integer p such that

$$N^p\{(K_i + I)/I\} \subset (K_p + I)/I$$

and

$$N^p(K_i \cap I) \subset K_p \cap I$$

for all positive integer i. Taking any positive integer j and any $r_1, r_2, \ldots, r_p$ in N, for each element x in K_{p+j}, we obtain that

$$r_1 r_2 \cdots r_p(x + I) \subset N^p(K_{p+j} + I) \subset K_p + I$$

for some $k \in K_p$ and a in I, $r_1 r_2 \cdots r_p x = k + a$. But $a \in K_{p+j}$ from this $a \in K_{p+j} \cap I$. Thus

$$N^p a \subset N^p(K_{p+j} \cap I) \subset K_{p+j} \cap I \subset K_p \cap I.$$

Finally we conclude that,

$$N^p r_1 r_2 \cdots r_p x = N^p(k+a) \subset N^p k + N^p a \subset K_p.$$

Thus $N^{2p} \subset K_p$, namely, $N^{2p}K_{p+j} \subset K_p$. Since j is arbitrary positive integer, this inclusion implies $N^{2p}K_i \subset K_p$ for all positive integer $i > p$. From proposition 2.2, N satisfies the $GACC$ on ideals. □

COROLLARY 2.5. *Let N be a left near-ring.*

(1) *A finite direct sum of ideals has the $GDCC$ on ideals if and only if each summand has the $GDCC$ on its ideals.*
(2) *A finite direct sum of ideals has the $GACC$ on ideals if and only if each summand has the $GACC$ on its ideals.*

A near-ring N is called *S-unital* if $a \in Na$ for each a in N and N is called *left bipotent* if $Na = Na^2$ for all a in N. For examples, every near-ring with left identity is S-unital and any Boolean near-ring is a left bipotent near-ring. We note that for any ideal (resp. N-subgroup) I of an S-unital near-ring N, we can construct that $I = NI = N^2I = N^3 = \cdots$.

LEMMA 2.6. *Let N be an arbitrary S-unital near-ring. Then we have:*

(1) *N satisfies the $GDCC$ on ideals (resp. N-subgroup) if and only if N satisfies the DCC on ideals (resp. N-subgroup).*
(2) *N satisfies the $GACC$ on ideals (resp. N-subgroup) if and only if N satisfies the ACC on ideals (resp. N-subgroup).*

Proof. (1) Suppose that N has the $GDCC$ on ideals. If $M_1 \supset M_2 \supset M_3 \supset \cdots$ is a descending chain of ideals of N, then there exists a positive integer p such that $N^pM_p \subset M_i$ for all positive integer i. Thus we conclude that

$$M_p = NM_p = N^2M_p = \cdots = N^pM_p \subset M_i,$$

for all positive integer i, that is, there exists a positive integer p such that

$$M_p = M_{p+1} = \cdots$$

Therefore N has the DCC for ideals. The converse statement is proved by the previous remark. Similary we have (2). □

A near-ring N is called *left K-regular* if for every element a in N there exists an element x in N and some positive integer n such that $a^n = xa^{n+1}$, similarly for right K-regular. These concepts are more general concepts of left regularity and right regularity of near-ring (or ring), N is said to be *generalized left bipotent* (*abbr. GLB*) if for any element a in N, there exists a positive integer n, such that $Na^n = Na^{n+1}$, similarly, for GBR. These concepts are generalized notion of left bipotent and right bipotent.

For each a in N, Na is called a *principal N-subgroup* of N, and $(0:a) = \{x \in N | xa = 0\}$ is denoted the (left) annihilator of a in N which is called a ***principal annihilator left ideal of N relative to a***. We see that $Na \supset Na^2 \supset Na^3 \supset \cdots$ is a descending chain of principal N-subgroups of N and $(0:a) \subset (0:a^2) \subset (0:a^3) \subset \cdots$ is an ascending chain of principal annihilator left ideals of N.

THEOREM 2.7. *Let N be any S-unital near-ring. Then the following statements are equivalent :*

(1) *N is left K-regular.*
(2) *N satisfies the DCC for principal N-subgroups of N.*
(3) *N satisfies the $GDCC$ for principal N-subgroups of N.*
(4) *N is GLB.*

Proof. (2)$\Longleftrightarrow$(3) is proved by Lemma 2.6. We shall onlly show that (1)$\Longrightarrow$(2). The remainder implications are left to the readers. Assume that N is left K-regular. Let a in N and consider

$$Na \supset Na^2 \supset Na^3 \supset \cdots$$

is a descending chain of principal N-subgroups of N. Since N is left K-regular there exists an element x in N and a positive integer n such that $a^n = xa^{n+1}$. On the other hand

$$Na^n = Nxa^{n+1} \subset Na^{n+1} = Naa^n \subset Na^n.$$

Thus we obtain the equality $Na^n = Na^{n+1}$. Using the similar method continuously, we have the following :

$$Na^{n+1} = Na^{n+2} = Na^{n+3} = \cdots .$$

Therefore, N satisfies the DCC for principal N-subgroups of N. □

3. Near-Rings with Associative-Derivations and Nil-Derivations

For any subset S of N, we write $Z(S)$ the center of S and put A is a non-zero N-subgroup of N. We now introduce the following special near-rings which are well known facts in [16] : A near-ring is *reduced* if N has no non-zero nilpotent elements, is *prime* if $a, b \in N$ and $aNb = 0$ implies $a = 0$ or $b = 0$ and has the *insertion of factors property* (*abbr. IFP*) provided that $ab = 0$ implies $axb = 0$ for all x in N.

LEMMA 3.1. *Let N be any near-ring.*

(1) *If N is reduced then N has the IFP.*
(2) *N has the IFP if and only if for any a in $N, (0 : a)$ is ideal of N if and only if for any $S \subset N, (0 : S)$ is an ideal of N.*

A derivation D on N is said to be *nilpotent* if there is a positive integer n such that $D^n(a) = 0$ for all a in N. The least such number is called the *index* of nilpotency of D, denoted by $n = nil(D)$. D is said to be *nil* if for each a in N, there is a natural number n (depending on a) such that $D^n(a) = 0$. The least such number is called the index of nilpotency of D with respect to a, denoted by $\text{nil}(D, a)$. Obviously nilpotent derivation is nil, but not vice versa. The latter can be seen from the following example : Consider $R[x]$, the near-ring of polynomials over a commutative ring R. Let D be ordinary derivation : $D(x^n) = nx^{n-1}$. It is routine to see that D is nil but not nilpotent.

THEOREM 3.2. *Let D be a derivation on a reduced near-ring N. Then every annihilator ideal is invariant under D. Moreover, we have ascending chain of annihilator ideals taking D repeatedly.*

Proof. Let S be any subset of N. Consider $(0 : S)$ as annihilator ideal of N with respect to S. We must show that $D\{(0 : S)\} \subset (0 : S)$. In fact, let $x \in (0 : S)$, that is, $xs = 0$ for all s in S. Taking D, we have $0 = D(xs) = D(x)s + sD(x)$. Multiplying s to the right side of this equality, $0 = D(x)s^2 + xD(s)s$. Since N is reduced by Lemma 3.1 (1), N has the IFP, so that $xD(s)s = 0$. Hence we get $D(x)s^2 = 0$ for all s in S. Again multiplying $D(x)$ to the left side of $D(x)s^2 = 0, (D(x)s)^2 = 0$. Because N is reduced we see that $D(x)s = 0$ for all s in S. Therefore $D(x) \in (0 : S)$. Moreover, let $xs = 0$ for all s in S. Taking derivation $D, 0 = D(xs) = D(x)s + xD(s)$. Multiplying x to the left side of this equality, $0 = xD(x) + x^2D(s)$. Since N is reduced, we apply lemma 3.1, so we get $xD(x)s = 0$ and $x^2D(s) = 0$. From this latter equality we get $xD(s) = 0$ for all s in S by using reducidility and multiplying $D(s)$ to the right side. Consequently we obtain $xD(s) = 0$. Hence $x \in (0 : D(S))$. The proof of this last statement is that $(0 : D(S)) \subset (0 : D^2(S))$.

Next we put $D(S) = S'$ and again, taking the above procedure to S' we obtain the inclusion $(0 : S') \subset (0 : D(S'))$, that is to say, $(0 : D(S)) \subset (0 : D^2(S))$ if we can repeat this procedure continuously, we have the following ascending chain of annihilator ideals of N :

$$(0 : S) \subset (0 : D(S)) \subset (0 : D^2(S)) \subset \cdots$$

In particular for any a in N we obtain that

$$(0 : a) \subset (0 : D(a)) \subset (0 : D^2(a)) \subset \cdots$$

as ascending chain of principal annihilator ideals of N by D taking repeatdly. □

From the Theorem 3.2, we have the following property of nil derivation.

COROLLARY 3.3. *Let D be a nil derivation on a reduced near-ring N. Then N has the ACC on principal annihilator ideals of N by taking D repeatedly.*

LEMMA 3.4. *Let D be a derivation on N. Then N satisfies the following distributive law*

$$a(D(bc)) = a(D(b)c + bD(c)) = aD(b)c + abD(c)$$

for all a, b, c in N.

Proof. From the expression for $D((ab)c) = D(a(bc))$, we have the result. □

LEMMA 3.5. *Let D be any derivation on N. If a in A is not a right zero divisor and a is commuting or skew commuting element then for any element b in $N, (a, b)$ is constant.*

Proof. From the equality $(a + b)a = a^2 + ba$, we obtain

$$D(a)a + D(b)a + aD(a) + bD(a) = D(a)a + aD(a) + D(b)a + bD(a).$$

First for a is commuting this equation reduces to

$$\{D(a) + D(b) - D(a) - D(b)\} = 0,$$

that is, $D(a + b - a - b)a = 0$. Since a is not a right zero divisor $D(a, b) = D(a + b - a - b) = 0$.

Second for a is skew commuting, since $D(a)a = -aD(a)$, above equation reduces to

$$D(b)a - D(a)a = -D(a)a + D(b)a,$$

that is, $D(a + b - a - b) = 0$. Because a is not a right zero divisor $D(a, b) = D(a + b - a - b) = 0$. Therefore, in the above both case, we see that (a, b) is constant. □

PROPOSITION 3.5. *Suppose a near-ring N has no non-zero divisors of zero. If N has a nontrivial commuting or skew commuting derivation D, then N becomes an abelian near-ring.*

Proof. Let c be any additive commutator in N. Then c is a constant by Lemma 3.4. Moreover, for any x in N, cx is also an additive commutator, hence also a constant. Thus we have that $0 = D(cx) = D(c)x + cD(x)$ and $cD(x) = 0$. Since D is nontrivial, there exists y in N such that $D(y) \neq 0$. So we see that $cD(y) = 0$. Applying N as no nonzero divisors of zero we conclude that $c = 0$. Hence N is abelian. □

THEOREM 3.6. *Let N be a prime near-ring with IFP.*

(1) *If $(A, +)$ is abelian, then N is an abelian near-ring.*
(2) *If $(A, \cdot)$ is commutative, then N is a commutative near-ring.*

Proof. (1) Let x, y be in N and a in A. Since A is an additive abelian N-subgroup we have the following equality :

$$xa + xa + ya + ya = xa + ya + xa + ya.$$

After we take the left and right cancellation, this equation becomes $(x, y)a = (x+y-x-y)a = 0$. From the fact N has the IFP, $(x, y)Na = 0$. Taking a is a nonzero element of A and using the fact N is prime, then $(x, y) = 0$. Whence N is an abelian near-ring.

(2) Let x, y be any elements in N and take a and b are both nonzero element of A. Since A is a commutative N-subgroup, we obtain the following equations :

$$[x,y]ab = xyab - yxab = xyab - ybxa = xayb - ybxa = ybxa - ybxa = 0.$$

From the fact N has the IFP, $[x,y]aNb = 0$. We can apply that N is prime, then $[x,y]a = 0$. Again from the fact that N is the IFP, we have $[x,y]Na = 0$, and again applying the property N is prime, consequently we get $[x,y] = 0$. Therefore N is commutative. □

We note that in Theorem 3.6. for any prime near-ring N with IFP, if A is nonzero additive abelian commutative N-subgroup, then N becomes a commutative ring.

A near-ring N is said to be *left strongly prime* if every non-zero left ideal of N contains a finite set whose left annihilator ideal is zero. Reduced prime Goldie near-rings and reduced prime near-ring with DCC on left annihilators are examples of left strongly prime near-rings. An ideal I of N satisfying $D(I) \subset I$ is called a *D-ideal.*

LEMMA 3.7. *Let D be a derivation on a near-ring N. Then, for each positive integer k, there exist integers $c_{k,i} (0 \leq i \leq k)$ such that*

$$D^k(x)y = \sum_{i=0}^{k} c_{k,i} D^{k-1}(xD^i(y))$$

i=0 for all x, y in N.

Proof. We proceed by induction on k. Obviously, $D(x)y = D(xy) - xD(y)$. Hence, we can put $c_{1,0} = 1$ and $c_{1,1} = -1$. Since

$$D^k(x)y = D(D^{k-1}(x))y = D(D^{k-1}(x)y) - D^{k-1}(x)D(y),$$

we can choose $c_{k,i}$ as follows that

$c_{k,0} = c_{k-1,0} = 1, c_{k,k} = -c_{k-1,k-1} = (-1)^k$ and $c_{k,i} = c_{k-1,i} - c_{k-1,i-1}$ $(0 < i < k)$. Consequently we see that

$$D^k(x)y = \sum_{i=0}^{k} c_{k,i} D^{k-1}(xD^i(y))$$

for all x, y in N. □

PROPOSITION 3.8. *Let N be a left strongly prme near-ring and D a derivation on N. If D is nil on a non-zero D-left ideal of N then D is a nil derivation on N.*

Proof. Let I be a non-zero D-left ideal of N. By hypothesis, I contains a finite set F with $(0 : F) = \{0\}$. There exists a positive integer m such that $D^m(x) = 0$ for all x in F. Let a be an arbitrary element of N. Then we can choose a positive integer n such that $D^n(aD^i(x)) = 0$ for all $i (0 \leq i \leq m-1)$ and all x in F. Then, by Lemma 3.7, we have $D^{m+n-1}(a)x = 0$ for all x in F, and therefore $D^{m+n-1}(a)F = 0$. Since $(0 : F) = \{0\}$, we conclude that $D^{m+n-1}(a) = 0$, for all a in N, which proves that D is nil on N. □

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