Pusan Kyŏngnam Math. J. 11(1995), No. 2, pp. 143–152

ON COMPACT SASAKIAN MANIFOLDS WITH VANISHING CONSTANT C–BOCHNER CURVATURE TENSOR

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1. Introduction

In a Sasakian manifold, many subjects for vanishing C-Bochner curvature tensor have been studied in [2]~[7], [9] and so on. Two of those, done by Hasegawa and Nakane [6] and Choi and Ki [3], assert the followings:

THEOREM H-N. *Let M be a n-dimensional Sasakian manifold with constant scalar curvature whose C-Bochner curvature tensor vanishes. If the square of the length $T_{(2)}$ of the ξ-Einstein tensor satisfies*

$$T_{(2)} \leq \frac{(n-3)(n+3)^2}{2(n-1)(n+1)^2(n-5)^2}(R+n-1)^2, \quad n \geq 7,$$

then M is a space of constant ϕ-holomorphic sectional curvature, and the equality is the best possible.

THEOREM C-K. *Let M be an $n(\geq 5)$-dimensional Sasakian manifold with constant scalar curvature whose C-Bochner curvature tensor vanishes. Then M is a space of constant ϕ-holomorphic sectional curvature $\frac{4R-(n-1)(3n-1)}{(n^2-1)}$ or M admits a cyclic parallel almost product structure which is not integrable.*

The purposes of this paper are to prove the following two theorems.

Received August 11, 1995.

This work was supported by a grant of the Basic Science Research Institute, Dong-A University, 1994 .

THEOREM 1. *Let M be an $n(\geq 5)$-dimensional compact Sasakian manifold with vanishing C-Bochner curvature tensor. If the length of the η-Einstein tensor $\|T_{(2)}\|$ satisfies*

$$(*) \qquad \|T_{(2)}\| \leq \frac{\sqrt{n(n-1)}(n+3)}{(n^2-1)(n-2)}(1-n-R),$$

then we have the same conclusion as that of Theorem C-K.

THEOREM 2. *Let M be an $n(\geq 7)$-dimensional Sasakian manifold with vanishing C-Bochner curvature tensor. If it satisfies*

$$(**) \qquad T_{(2)} = a(R+n-1)^2$$

for some constant a, then we have the same conclusion as that of Theorem C-K.

2. Preliminaries

Let M be an $n(> 3)$-dimensional Sasakian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$, where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, n\}$(The summation convention will be used with respect to these indices). If we denote by ∇ the operator of covariant differentiation with respect to the Riemannian connection of M, then there exists a unit Killing vector ξ^h satisfying

$$(2.1) \qquad \begin{cases} \phi_j{}^r\phi_r{}^h = -\delta_j{}^h + \eta_j\xi^h, \quad \eta_j = g_{jr}\xi^r, \quad \eta_r\phi_j{}^r = 0, \\ \phi_r{}^h\xi^r = 0, \quad g_{rs}\phi_j{}^r\phi_i{}^s = g_{ji} - \eta_j\eta_i, \quad \phi_{ji} + \phi_{ij} = 0, \end{cases}$$

$$(2.2) \qquad \phi_{ji} = \nabla_j\eta_i, \quad \nabla_k\phi_{ji} = -g_{kj}\eta_i + g_{ki}\eta_j.$$

Because of the Ricci formula for ξ^i, it is clear that

$$(2.3) \qquad R_{kji}{}^r\eta_r = \eta_k g_{ji} - \eta_j g_{ki}$$

and hence

$$(2.4) \qquad R_{jr}\xi^r = (n-1)\eta_j,$$

where R_{kjih} and R_{ji} denote the components of the Riemannian curvature tensor K and of the Ricci tensor Ric respectively.

It is well known that in a Sasakian manifold the following equations hold:

$$H_{ji} + H_{ij} = 0, \tag{2.5}$$

$$R_{ji} = R_{rs}\phi_j{}^r\phi_i{}^s + (n-1)\eta_j\eta_i, \tag{2.6}$$

$$\begin{aligned}\nabla_k R_{ji} - \nabla_j R_{ki} = (\nabla_s R_{kr})\phi_j{}^r\phi_i{}^s - \{H_{ki} - (n-1)\phi_{ki}\}\eta_j \\ - 2\{H_{jk} - (n-1)\phi_{kj}\}\eta_i,\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\nabla_k R_{ji} - (\nabla_k R_{rs})\phi_j{}^r\phi_i{}^s \\ &= -\{H_{kj} - (n-1)\phi_{kj}\}\eta_i - \{H_{ki} - (n-1)\phi_{ki}\}\eta_j,\end{aligned} \tag{2.8}$$

$$\xi^r\nabla_r R_{kji}{}^h = 0, \tag{2.9}$$

where we put $H_{ji} = \phi_j{}^r R_{ri}$.

We denote a tensor field $W^{(m)}$ with components $W_{ji}^{(m)}$ and a function $W_{(m)}$ for any positive integer m as follows:

$$\begin{aligned}W_{ji}^{(m)} &= W_{ji_1}W_{i_2}{}^{i_1}\cdots W_i{}^{i_{m-1}}, \\ W_{(m)} &= TrW^{(m)} = \sum_i W_{ii}^{(m)}.\end{aligned} \tag{2.10}$$

Also, we define the η-Einstein tensor T_{ji} by

$$T_{ji} = R_{ji} - (\frac{R}{n-1} - 1)g_{ji} + (\frac{R}{n-1} - n)\eta_j\eta_i. \tag{2.11}$$

If the η-Einstein tensor T vanishes, then M is called an η-Einstein manifold. From (2.4) and (2.5), we have

$$\mathrm{Tr}T = T_r{}^r = 0, \tag{2.12}$$

$$T_{jr}\xi^r = 0, \tag{2.13}$$

$$T^{(m)}_{jr}\phi_i{}^r + T^{(m)}_{ir}\phi_j{}^r = 0 \tag{2.14}$$

for any integer m.

A Sasakian manifold M is called a space of constant ϕ-holomorphic sectional curvature c if the curvature tensor of M has the form:

$$R_{kji}{}^h = \frac{c+3}{4}(g_{ji}\delta_k{}^h - g_{ki}\delta_j{}^h) + \frac{c-1}{4}(g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h + \eta_k\eta_i\delta_j{}^h - \eta_j\eta_i\delta_k{}^h - \phi_{ki}\phi_j{}^h + \phi_{ji}\phi_k{}^h - 2\phi_{kj}\phi_i{}^h).$$

Matsumoto and Chūman ([8]) introduced the C-Bochner curvature tensor $B_{kji}{}^h$ defined by

$$\begin{aligned} B_{kji}{}^h =\ & R_{kji}{}^h + \frac{1}{n+3}(R_{ki}\delta_j{}^h - R_{ji}\delta_k{}^h + g_{ki}R_j{}^h \\ & - g_{ji}R_k{}^h + H_{ki}\phi_j{}^h - H_{ji}\phi_k{}^h + \phi_{ki}H_j{}^h - \phi_{ji}H_k{}^h + 2H_{kj}\phi_i{}^h \\ & + 2\phi_{kj}H_i{}^h - R_{ki}\eta_j\xi^h + R_{ji}\eta_k\xi^h - \eta_k\eta_iR_j{}^h + \eta_j\eta_iR_k{}^h) \\ & - \frac{k+n-1}{n+3}(\phi_{ki}\phi_j{}^h - \phi_{ji}\phi_k{}^h + 2\phi_{kj}\phi_i{}^h) \\ & - \frac{k-4}{n+3}(g_{ki}\delta_j{}^h - g_{ji}\delta_k{}^h) \\ & + \frac{k}{n+3}(g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h + \eta_k\eta_i\delta_j{}^h - \eta_j\eta_i\delta_k{}^h), \end{aligned} \tag{2.15}$$

where $k = \frac{R+n-1}{n+1}$. It is well-known that if a Sasakian manifold with vanishing C-Bochner curvature tensor is an η-Einstein manifold, then it is a space of constant ϕ-holomorphic sectional curvature.

By a straightforward computation, we can prove

$$\begin{aligned} \frac{n+3}{n-1}\nabla_rB_{kji}{}^r =\ & \nabla_kR_{ji} - \nabla_jR_{ki} - \eta_k\{H_{ji} - (n-1)\phi_{ji}\} \\ & + \eta_j\{H_{ki} - (n-1)\phi_{ki}\} + 2\eta_i\{H_{kj} - (n-1)\phi_{kj}\} \\ & + \frac{1}{2(n+1)}\{(g_{ki} - \eta_k\eta_i)\delta_j{}^r - (g_{ji} - \eta_j\eta_i)\delta_k{}^r \\ & + \phi_{ki}\phi_j{}^r - \phi_{ji}\phi_k{}^r + 2\phi_{kj}\phi_i{}^r\}R_r, \end{aligned} \tag{2.16}$$

where we put $R_j = \nabla_jR$.

3. Vanishing C-Bochner Curvature Tensor

Let M be an $n(\geq 5)$-dimensional Sasakian manifold with vanishing C-Bochner curvature tensor. By (2.1), (2.4), (2.7)~(2.9) and (2.16), we then obtain

$$
\begin{aligned}
\nabla_k R_{ji} = {} & \{R_{kr} - (n-1)g_{kr}\}(\phi_j{}^r\eta_i + \phi_i{}^r\eta_j) \\
& + \frac{1}{2(n+1)}\{2R_k(g_{ji} - \eta_j\eta_i) + R_j(g_{ki} - \eta_k\eta_i) \\
& + R_i(g_{kj} - \eta_k\eta_j) - \phi_{kj}w_i - \phi_{ki}w_j\},
\end{aligned} \tag{3.1}
$$

where we put $w_j = \phi_{jr}R^r$.

Operating $g^{jl}\nabla_l$ to (3.1) and taking account of (2.1), (2.2), (2.5) and (3.1), we have

$$
\begin{aligned}
g^{jl}\nabla_l\nabla_k R_{ji} = {} & R_{ki} - (n-1)g_{ki} - \{R - n + 1\}\eta_k\eta_i \\
& + (n-1)^2\eta_k\eta_i - \frac{1}{2}w_k\eta_j + \frac{1}{2(n+1)}\{3(\nabla_k R_i + w_k\eta_i) \\
& + \Delta R(g_{ki} - \eta_k\eta_i) - (n-2)w_i\eta_k - (\nabla_r R_s)\phi_k{}^r\phi_i{}^s\},
\end{aligned} \tag{3.2}
$$

where $\Delta R = g^{ji}\nabla_j R_i$.

On the other hand we have the Ricci identity for R_{ji}

$$
\nabla^l\nabla_j R_{li} - \frac{1}{2}\nabla_j R_i = R^{(2)}_{ji} - R_{kjih}R^{kh}.
$$

Thus, (3.2) is reduced to

$$
\begin{aligned}
& (n+3)R^{(2)}_{ji} - (n+3)R_{kjih}R^{kh} \\
& = (n+3)R_{ji} - (n+3)(n-1)g_{ji} \\
& - (n+3)\{R - n(n-1)\}\eta_j\eta_i \\
& + \frac{n+3}{2(n+1)}\{\Delta R(g_{ji} - \eta_j\eta_i) + \nabla_r R_s\phi_j{}^r\phi_i{}^s\} \\
& - \frac{(n+3)(n-2)}{2(n+1)}(\nabla_j R_i + w_j\eta_i + w_i\eta_j).
\end{aligned} \tag{3.3}
$$

Using (2.15), we obtain (see [7])

$$\begin{aligned}
&(n+3)R_{kjih}R^{kh} \\
&= 4R^{(2)}_{ji} - (4n - R + 2k)R_{ji} + \{R_{(2)} - (k-4)R + (n-1)k\}g_{ji} \\
&\quad - \{R_{(2)} + (n-1)^2 - (n-1)k - kR\}\eta_j\eta_i.
\end{aligned}$$

Combining this with (3.3), we have

$$\begin{aligned}
&R^{(2)}_{ji} - \beta R_{ji} - \gamma g_{ji} - \{(n-1)^2 - (n-1)\beta - \gamma\}\eta_j\eta_i \\
&= \frac{n+3}{2(n^2-1)}\{\Delta R(g_{ji} - \eta_j\eta_i) - \nabla_r R_s \phi_j{}^r \phi_i{}^s\} \\
&\quad - \frac{(n+3)(n-2)}{2(n^2-1)}(\nabla_j R_i + w_j\eta_i + w_i\eta_j),
\end{aligned} \tag{3.4}$$

where we have defined

$$\begin{aligned}
(n+1)\beta &= R - 3n - 5, \\
(n-1)\gamma &= R_{(2)} - \frac{1}{n+1}R^2 + 4R - \frac{n-1}{n+1}(n^2 + 3n + 4).
\end{aligned}$$

Transforming (2.11) by $R_k{}^t$ and taking account of (2.4) and (3.4), we can get

$$\begin{aligned}
T_{jr}R_k{}^r &= (\beta + 1 - \frac{R}{n-1})R_{jk} + \gamma g_{jk} \\
&+ \{R - n + 1 - (n-1)\beta - \gamma\}\eta_j\eta_k \\
&+ \frac{(n+3)}{2(n^2-1)}\{\Delta R(g_{jk} - \eta_j\eta_k) - \nabla_r R_s \phi_j{}^r \phi_k{}^s\} \\
&- \frac{(n+3)(n-2)}{2(n^2-1)}(\nabla_j R_k + w_j\eta_k + w_k\eta_j),
\end{aligned}$$

which together with (2.11) and (2.13) yields

$$\begin{aligned}
T^{(2)}_{ji} &= -\frac{n+3}{n^2-1}(R + n - 1)T_{ji} + \frac{T_{(2)}}{n-1}(g_{ji} - \eta_j\eta_i) \\
&+ \frac{n+3}{2(n^2-1)}\{\Delta R(g_{ji} - \eta_j\eta_i) + \nabla_r R_s \phi_j{}^r \phi_i{}^s\} \\
&- \frac{(n+3)(n-2)}{2(n^2-1)}(\nabla_j R_i + w_j\eta_i + w_i\eta_j).
\end{aligned} \tag{3.5}$$

By using (2.14), we can easily, taking account of (3.5), see that

$$\phi_{jr}\nabla_i R^r + \phi_{ir}\nabla_j R^r = R_j\eta_i + R_i\eta_j,$$

which implies

$$\nabla_r R_s \phi_j{}^r \phi_i{}^s = \nabla_j R_i + w_j\eta_i + w_i\eta_j.$$

Thus, (3.5) turns out to be

$$\begin{aligned}(3.6)\quad T^{(2)}_{ji} =& -\frac{n+3}{n^2-1}(R+n-1)T_{ji} + (\frac{1}{n-1}T_{(2)} \\ &+ \frac{n+3}{2(n^2-1)}\Delta R)(g_{ji} - \eta_j\eta_i) - \frac{n+3}{2(n+1)}(\nabla_j R_i + w_j\eta_i + w_i\eta_j).\end{aligned}$$

Accordingly, we see, using (2.12) and (2.13), that

$$T_{(3)} + \frac{n+3}{n^2-1}(R+n-1)T_{(2)} + \frac{n+3}{2(n+1)}\nabla_j R_i T^{ji} = 0.$$

By the way, using (2.11) and the fact that $\nabla_j R_i{}^j = \frac{1}{2}R_i$, we have

$$\nabla^j(T_{ji}R^i) = \frac{n-3}{2(n-1)}\|R_i\|^2 + \nabla_j R_i T^{ji}.$$

Combining the last two equations, it follows that

$$\begin{aligned}(3.7)\quad &\frac{n^2-1}{n+3}\nabla^j(T_{ji}R^i) \\ &= \frac{n^2-1}{n+3}T_{(3)} + (R+n-1)T_{(2)} - \frac{n-3}{4}\|R_i\|^2.\end{aligned}$$

Proof of Theorem 1. Let $c_1, \cdots, c_m$ and k be real numbers satisfying $\sum_{i=1}^m c_i = 0$ and $\sum_{i=1}^m c_i^2 = k^2$. Then we have [9]

$$(3.8)\quad -\frac{m-2}{\sqrt{m(m-1)}}k^3 \le \sum_{i=1}^m c_i^3 \le \frac{m-2}{\sqrt{m(m-1)}}k^3.$$

From (2.12) and (2.13) and commutativity of $R_j{}^h$ and $\phi_j{}^h$, we see that the eigenvalues of $R_j{}^h$ are $c_1, \cdots, c_{n/2}, c_1, \cdots, c_{n/2}$ and 0. Combining this fact with (3.8), we have

$$\frac{n^2-1}{n+3}T_{(3)} + (R+n-1)T_{(2)} \le T_{(2)}\{\frac{n-2}{\sqrt{n(n-1)}}\|T_{(2)}\| + R + n - 1\}.$$

Because of the condition (*), the right hand side of (3.7) is nonnegative. M being compact, using the Green's theorem, we see that R is constant. Owing to Theorem C-K, the conclusion of Theorem 1 is true. This completes the proof. □

Proof of Theorem 2. Differentiating (2.11) covariantly, we find

$$\begin{aligned} \nabla_k T_{ji} &= \nabla_k R_{ji} - \frac{1}{n-1}R_k(g_{ji} - \eta_j\eta_i) \\ &+ (\frac{R}{n-1} - n)(\phi_{kj}\eta_i + \phi_{ki}\eta_j). \end{aligned} \tag{3.9}$$

If we transvect T^{ji} to this and take account of (2.12), (2.13) and (3.1), then we obtain

$$T^{ji}\nabla_k T_{ji} = \frac{2}{n-1}T_{kr}R^r,$$

which implies

$$\nabla_j T_{(2)} = \frac{4}{n-1}T_{kr}R^r.$$

Because of the condition (**), it is seen that

$$T_{jr}R^r = \frac{n-1}{2}a(R+n-1)R_j. \tag{3.10}$$

Differentiating (3.10) covariantly and making use of (3.9), we find

$$\begin{aligned} &(\nabla_k R_{jr})R^r + (\frac{R}{n-1} - n)w_k\eta_j \\ &= \frac{1}{n-1}R_jR_k - T_{jr}\nabla_k R^r + \frac{n-1}{2}a\{R_jR_k + (R+n-1)\nabla_k R_j\}, \end{aligned}$$

which together with (3.1) implies that

$$\begin{aligned}
&-\{R_{kr}w^r-(n-1)w_k\}\eta_j+(\frac{R}{n-1}-n)w_k\eta_j\\
&\quad+\frac{1}{2(n+1)}\{3R_jR_k+\|R_i\|^2(g_{jk}-\eta_j\eta_k)-w_jw_k\}\\
&=\frac{1}{n-1}R_jR_k-T_{jr}\nabla_kR^r+\frac{n-1}{2}a\{R_jR_k+(R+n-1)\nabla_kR_j\}.
\end{aligned}$$

By transvecting η^j, we obtain

$$\begin{aligned}
&-\{R_{kr}w^r-(n-1)w_k\}+(\frac{R}{n-1}-n)w_k\\
&=-\frac{n-1}{2}a(R+n-1)w_k,
\end{aligned}$$

where we have used (2.1) and (2.9). From the last two equations it follows that

$$\begin{aligned}
&\frac{1}{2(n+1)}\{3R_jR_k-w_jw_k+\|R_i\|^2(g_{jk}-\eta_j\eta_k)\}\\
&-\frac{n-1}{2}a(R+n-1)w_k\eta_j\\
&=\frac{1}{n-1}R_jR_k-T_{jr}\nabla_kR^r+\frac{n-1}{2}a\{R_jR_k+(R+n-1)\nabla_kR_j\}.
\end{aligned}$$

Transvecting this with R^j and making use of (3.10), we see that $R_j=0$, namely R is constant. Therefore we arrive at the conclusion. □

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