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SOME HOMOMORPHISMS OF Γ-RINGS

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DEFINITION 1. *([2]) Let M and Γ be additive abelians groups. If we have two maps $M \times \Gamma \times M \to M$ and $\Gamma \times M \times \Gamma \to \Gamma$ such that for all $x, y, z \in M$, $\alpha, \beta, \gamma \in \Gamma$,*

(1) $(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z)$ and $(\alpha x\beta)y\gamma = \alpha(x\beta y)\gamma = \alpha x(\beta y\gamma)$,

(2) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)z = x\alpha z + x\beta z$, $x\alpha(y+z) = x\alpha y + x\alpha z, (\alpha+\beta)x\gamma = \alpha x\gamma + \beta x\gamma, \alpha(x+y)\gamma = \alpha x\gamma + \alpha y\gamma$ and $\alpha x(\beta+\gamma) = \alpha x\beta + \alpha x\gamma$.

then M is called a weak Γ-ring in the sense of Nobusawa.

In this paper M denotes a weak Nobusawa Γ-ring. For the definitions of basic notions in Γ-rings, we refer to [3,4].

DEFINITION 2. *([3]) A pair of mappings (θ, ϕ) from Γ_1-ring M_1 onto Γ_2-ring M_2 is called a homomorphism if it satisfies the following properties:*

(1) *θ and ϕ are group homomorphisms from M_1 onto M_2 and from Γ_1 onto Γ_2 respectively.*

(2) *$\theta(x\alpha y) = \theta(x)\phi(\alpha)\theta(y)$, $\phi(\alpha x\beta) = \phi(\alpha)\theta(x)\phi(\beta)$ for all $x, y \in M_1, \alpha, \beta \in \Gamma_1$.*

If θ, ϕ are both injective , then (θ, ϕ) is said to be an isomorphism of Γ_1-ring M_1 onto Γ_2-ring M_2.

THEOREM 3. *Let M_1 be a Γ_1-ring and A, B two abelian groups. Suppose that there exist two mappings $M_1 \times A \to B, (x,u) \mapsto xu$ and $B \times \Gamma_1 \to A, (v, \alpha) \mapsto v\alpha$ satisfying*

$$x((yu)\alpha) = (x\alpha y)u, (x(v\beta))\alpha = v(\alpha x\beta),$$

$$(x+y)u_1 = xu_1 + yu_1, x(u_1+u_2) = xu_1 + xu_2,$$

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$$v(\alpha+\beta) = v\alpha + v\beta$$

and

$$(v_1+v_2)\alpha = v_1\alpha + v_2\alpha$$

for all $x, y \in M_1$, $\alpha, \beta \in \Gamma_1, u_1, u_2 \in A$ and $v_1, v_2 \in B$, and $xu = yu$ for $x, y \in M_1$ and for all $u \in A$ implies $x = y$, and $v\alpha = v\beta$ for $\alpha, \beta \in \Gamma_1$ and for all $v \in B$ implies $\alpha = \beta$. Then there exists a Γ_2-ring M_2 such that Γ_1-ring M_1 is isomorphic to Γ_2-ring M_2, where each element of M_2 is a mapping from A to B and each element of Γ_2 is a mapping from B to A.

Proof. Let $x \in M_1$ and $\alpha \in \Gamma_1$. Define

$$l_x : A \to B$$

by

$$l_x(u) = xu$$

for all $u \in A$ and define

$$r_\alpha : B \to A$$

by

$$r_\alpha(v) = v\alpha$$

for all $v \in B$. Let $M_2 = \{l_x | x \in M_1\}$ and $\Gamma_2 = \{r_\alpha | \alpha \in \Gamma_1\}$. Then, since A and B are abelian groups, M_2 and Γ_2 are abelian groups under the pointwise addition. Define two mappings

$$M_2 \times \Gamma_2 \times M_2 \to M_2$$

by

$$(l_x, r_\alpha, l_y) \mapsto l_x \circ r_\alpha \circ l_y$$

and

$$\Gamma_2 \times M_2 \times \Gamma_2 \to \Gamma_2$$

by

$$(r_\alpha, l_x, r_\beta) \mapsto r_\alpha \circ l_x \circ r_\beta$$

the usual compositions of mappings. For any $u \in A$, we have that

$$\begin{aligned}(l_x \circ r_\alpha \circ l_y)(u) &= l_x \circ r_\alpha(l_y(u)) \\ &= l_x(r_\alpha(l_y(u))) \\ &= x((yu)\alpha) \\ &= (x\alpha y)u \\ &= l_{x\alpha y}(u).\end{aligned}$$

Hence

$$l_x \circ r_\alpha \circ l_y = l_{x\alpha y} \in M_2.$$

Next for any $v \in B$, we get the equality:

$$\begin{aligned}(r_\alpha \circ l_x \circ r_\beta)(v) &= r_\alpha \circ l_x(r_\beta(v)) \\ &= r_\alpha(l_x(r_\beta(v))) \\ &= (x(v\beta))\alpha \\ &= v(\alpha x\beta) \\ &= r_{\alpha x\beta}(v).\end{aligned}$$

Hence

$$r_\alpha \circ l_x \circ r_\beta = r_{\alpha x\beta} \in \Gamma_2.$$

Let $l_x, l_y, l_z \in M_2$ and $r_\alpha, r_\beta, r_\gamma \in \Gamma_2$. Then we have

$$(l_x + l_y) \circ r_\alpha \circ l_z = l_x \circ r_\alpha \circ l_z + l_y \circ r_\alpha \circ l_z,$$

$$l_x \circ (r_\alpha + r_\beta) \circ l_y = l_x \circ r_\alpha \circ l_y + l_x \circ r_\beta \circ l_y,$$

$$l_x \circ r_\alpha \circ (l_y + l_z) = l_x \circ r_\alpha \circ l_y + l_x \circ r_\alpha \circ l_z,$$

$$(r_\alpha + r_\beta) \circ l_x \circ r_x = r_\alpha \circ l_x \circ r_\gamma + r_\beta \circ l_x \circ r_\gamma,$$

$$r_\alpha \circ (l_x + l_y) \circ r_\beta = r_\alpha \circ l_x \circ r_\beta + r_\alpha \circ l_y \circ r_\beta$$

and

$$r_\alpha \circ l_x \circ (r_\beta + r_\gamma) = r_\alpha \circ l_x \circ r_\beta + r_\alpha \circ l_x \circ r_\gamma.$$

Clearly

$$l_x \circ r_\alpha \circ (l_y \circ r_\beta \circ l_z) = l_x \circ (r_\alpha \circ l_y \circ r_\beta) \circ l_z = (l_x \circ r_\alpha \circ l_y) \circ r_\beta \circ l_z$$

and

$$\begin{aligned}(r_\alpha \circ l_x \circ r_\beta) \circ l_y \circ r_\gamma &= r_\alpha \circ (l_x \circ r_\beta \circ l_y) \circ r_\alpha \\ &= r_\alpha \circ l_x \circ (r_\beta \circ l_y \circ r_\gamma).\end{aligned}$$

Hence M_2 is a Γ_2-ring. Now define $\theta : M_1 \to M_2, \phi : \Gamma_1 \to \Gamma_2$ by $\theta(x) = l_x$ and $\phi(\alpha) = r_\alpha$ for all $x \in M_1$ and all $\alpha \in \Gamma_1$ respectively. Then θ and ϕ are onto group homomorphisms. Indeed, for any $x, y \in M_1$ and $u \in A$,

$$\begin{aligned}\theta(x+y)(u) &= l_{x+y}(u) = (x+y)u \\ &= xu + yu = l_x(u) + l_y(u) \\ &= \theta(x)(u) + \theta(y)(u) \\ &= (\theta(x) + \theta(y))(u).\end{aligned}$$

Thus we have

$$\theta(x+y) = l_{x+y} = l_x + l_y = \theta(x) + \theta(y).$$

Also, for any $\alpha, \beta \in \Gamma_1$ and $v \in B$,

$$\begin{aligned}\phi(\alpha+\beta)(v) &= r_{\alpha+\beta}(v) \\ &= v(\alpha+\beta) \\ &= v\alpha + v\beta \\ &= r_\alpha(v) + r_\beta(v) \\ &= \phi(\alpha)(v) + \phi(\beta)(v) \\ &= (\phi(\alpha) + \phi(\beta))(v).\end{aligned}$$

Thus we obtain

$$\phi(\alpha+\beta) = \phi(\alpha) + \phi(\beta).$$

Clearly θ and ϕ are onto. For all $x, y \in M_1$ and $\alpha, \beta \in \Gamma_1$,

$$\theta(x\alpha y) = l_{x\alpha y} = l_x \circ r_\alpha \circ l_y = \theta(x)\phi(\alpha)\theta(y)$$

and

$$\phi(\alpha x\beta) = r_{\alpha x\beta} = r_\alpha \circ l_x \circ r_\beta = \phi(\alpha)\theta(x)\phi(\beta).$$

Thus (θ, ϕ) is a homomorphism from Γ_1-ring M_1 onto Γ_2-ring M_2. Suppose that $\theta(x) = \theta(y)$ for $x, y \in M_1$. Then $l_x = l_y$. Hence $l_x(u) = l_y(u)$ for all $u \in A$. This implies that $xu = yu$ for all $u \in A$. By the given conditions, we have $x = y$. Hence θ is injective. And suppose that $\phi(\alpha) = \phi(\beta)$ for $\alpha, \beta \in \Gamma_1$. Then $r_\alpha = r_\beta$ and so $r_\alpha(v) = r_\beta(v)$ for all $v \in B$. This implies that $v\alpha = v\beta$ for all $v \in B$. By the hypothesis, $\alpha = \beta$ and hence ϕ is injective. Thus (θ, ϕ) is an isomorphism from Γ_1-ring M_1 onto Γ_2-ring M_2. Therefore the Γ_1-ring M_1 is isomorphic to the Γ_2-ring M_2.

THEOREM 4. *Let A and B be two abelians groups. Then we can make a Γ-ring M with the following properties:*

(1) *There exist two mappings $M \times A \to B, (x, u) \mapsto xu$ and $B \times \Gamma \to A, (v, \alpha) \mapsto v\alpha$ such that $x((yu)\alpha) = (x\alpha y)u$ and $(x(v\beta))\alpha = v(\alpha x\beta)$ for $x, y \in M, \alpha, \beta \in \Gamma$ and $u \in A, v \in B$.*

(2) *$xu = yu$ for $x, y \in M$ and for all $u \in A$ implies $x = y$ and $v\alpha = v\beta$ for $\alpha, \beta \in \Gamma$ and for all $v \in B$ implies $\alpha = \beta$.*

Proof. Let M be the set of all homomorphisms from A to B and Γ the set of all homomorphisms from B to A. Then M and Γ are abelian groups. Define two maps $M \times \Gamma \times M \to M$, $(f, \alpha, g) \mapsto f \circ \alpha \circ g$ and $\Gamma \times M \times \Gamma \to \Gamma$, $(\alpha, f, \beta) \mapsto \alpha \circ f \circ \beta$ the usual composition of mappings. Let $f, g, h \in M$ and $\alpha, \beta, \gamma \in \Gamma$. For any $u \in A$,

$$(f \circ \alpha \circ g)(u) = f(\alpha(g(u))) \in B,$$

since $g(u) \in B$ and $\alpha(g(u)) \in A$. Thus $f \circ \alpha \circ g \in M$. Moreover,

$$(f \circ \alpha \circ g) \circ \beta \circ h = f \circ (\alpha \circ g \circ \beta) \circ h = f \circ \alpha \circ (g \circ \beta \circ h).$$

Since f, g, h, α, β and γ are homomorphisms, we have

$$(f+g) \circ \alpha \circ h = f \circ \alpha \circ h + g \circ \alpha \circ h, f \circ (\alpha + \beta) \circ h = f \circ \alpha \circ h + f \circ \beta \circ h$$

$$f \circ \alpha \circ (g + h) = f \circ \alpha \circ g + f \circ \alpha \circ h, (\alpha + \beta) \circ f \circ \gamma = \alpha \circ f \circ \gamma + \beta \circ f \circ \gamma,$$

and

$$\alpha\circ(f+g)\circ\gamma=\alpha\circ f\circ\gamma+\alpha\circ g\circ\gamma, \alpha\circ f\circ(\beta+\gamma)=\alpha\circ f\circ\beta+\alpha\circ f\circ\gamma.$$

Define two mappings $M\times A\to B$ by $(f,u)\mapsto f(u)$ and $B\times\Gamma\to A, (v,\alpha)\mapsto\alpha(v)$ for all $f\in M,\alpha\in\Gamma$ and $u\in A, v\in B$.

Now

$$g((fu)\alpha)=g(f(u)\alpha)=g(\alpha(f(u)))=(g\alpha f)(u),$$

$$(f(v\beta))\alpha=(f(\beta(v)))\alpha=\alpha(f(\beta(v)))=v(\alpha f\beta).$$

for $f,g\in M,\alpha,\beta\in\Gamma$ and $u\in A, v\in B$. Hence the condition (1) is satisfied. Let $fu=gu$ for $f,g\in M$ and for all $u\in A$. Then we have $f(u)=g(u)$ for all $u\in A$ and so $f=g$. Also $v\alpha=v\beta$ for $\alpha,\beta\in\Gamma$ and for all $v\in B$ implies $\alpha=\beta$. Thus the condition (2) is satisfied.

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