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GLOBAL REGULARITY OF THE $\overline{\partial}$–NEUMANN PROBLEM ON PSEUDOCONVEX COMPLEX MANIFOLDS

JINKEE LEE AND HONG RAE CHO

1. Introduction

Let X be a complex manifold of dimension n. Let $\Omega \Subset X$ be an open submanifold with smooth boundary. The $\overline{\partial}$-Neumann problem is concerned with the existence and especially with the regularity of the solution u of $\overline{\partial}u = \alpha$, where u is orthogonal to the kernel of $\overline{\partial}$ and α is a $\overline{\partial}$-closed (p,q)-form with L^2-coefficients and it is cohomologous to zero on Ω. One of the main methods for proving regularity of the solution is the method of subelliptic estimates. The importance of subelliptic estimates lies in the fact that it yields a positive answer to the question of local regularity: If the form α is smooth in a neighborhood U of a given boundary point z_0, is the solution u also smooth in U ? However, for many applications, such as the boundary regularity of biholomorphic maps, it is sufficient to study the question of global regularity: If α is smooth on all of $\overline{\Omega}$, is the solution u also smooth on all of $\overline{\Omega}$? It is not yet known whether the special solution, namely the one that is orthogonal to the kernel of $\overline{\partial}$, is smooth. However, Kohn and Nirenberg [5] found that the global regularity for the special solution does hold when a certain estimate, which we shall call a compactness estimate, still holds for the domain Ω. A compactness estimate is said to hold for the $\overline{\partial}$-Neumann problem on Ω if for every $\varepsilon > 0$, there is a function $\zeta_\varepsilon \in C_o^\infty(\Omega)$ such that

$$\|f\|^2 \leq \varepsilon Q(f,f) + \|\zeta_\varepsilon f\|_{-1}^2, \qquad f \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*).$$

Here $Q(f,f)$ refers to the form $(\overline{\partial}f,\ \overline{\partial}f)+(\overline{\partial}^* f,\ \overline{\partial}^* f)$, and $\|\cdot\|_{-1}$ refers to the Sobolev norm of order -1 for forms on Ω.

And we shall require the following definition.

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DEFINITION. The boundary of Ω satisfies property (P) at $z \in b\Omega$ if for every positive number M there is a plurisubharmonic function $\lambda \in C^\infty(\overline{\Omega})$ with $0 \leq \lambda \leq 1$, such that

$$\sum_{j,k=1}^{n} \lambda_{jk}(z) t_j \overline{t_k} \geq M|t|^2,$$

where $\lambda_{jk}(z)$, $j, k = 1, \cdots, n$, is defined by $\partial\overline{\partial}\lambda(z) = \sum_{j,k=1}^{n} \lambda_{jk}(z)\omega^j \wedge \overline{\omega}^k$ for an orthonormal basis $\omega^1, \cdots, \omega^n$ of $\Lambda_z^{1,0}$. We say that the boundary of Ω satisfies property (P) if it satisfies property (P) at each boundary point of Ω.

Catlin [2] showed that a compactness estimate holds for the $\overline{\partial}$-Neumann problem on a smoothly bounded pseudoconvex domain Ω in $\mathbb{C}^n$ which satisfies property (P). In this paper, we shall show the following case of the complex manifold.

THEOREM. *Let Ω be a smoothly bounded, pseudoconvex submanifold which is relatively compact in a complex manifold X. If $b\Omega$ satisfies property (P), then the compactness estimate holds for the $\overline{\partial}$-Neumann problem on Ω.*

We define

$$\mathcal{H}^{p,q} = \{\alpha \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*) \ ; \ \overline{\partial}\alpha = 0 \text{ and } \overline{\partial}^*\alpha = 0\}.$$

By the Kohn-Nirenberg theorem [5], we get the following corollary.

COROLLARY. *Let m be a nonnegative integer and $H_m(\Omega)$ be a Sobolev space of order m with the norm $\|\cdot\|_m$. Under the hypotheses of Theorem, if α is a $\overline{\partial}$-closed (p,q)-form, which is C^∞ on $\overline{\Omega}$ and $\alpha \perp \mathcal{H}^{p,q}$, then the canonical solution u of $\overline{\partial}u = \alpha$ with $u \perp Ker(\overline{\partial})$ satisfies $\|u\|_m^2 \leq C_m(\|\alpha\|_m^2 + \|u\|^2)$. Since $C^\infty(\overline{\Omega}) = \cap_{m=0}^{\infty} H_m(\Omega)$, it follows that if $\alpha \in C^\infty_{(p,q)}(\overline{\Omega})$, then $u \in C^\infty_{(p,q-1)}(\overline{\Omega})$.*

2. L^2-estimate for $\bar\partial$.

We shall use Hörmander's method of weighted estimates for $\bar\partial$. By the Gram-Schmidt process in a coordinate patch U, we can construct forms $\omega^1, \ldots, \omega^n$, which for all z are an orthonormal basis of $\Lambda_z^{1,0}(U)$. Furthermore we can choose $\omega^n = \sqrt{2}\,\partial\rho$ on $b\Omega$, where ρ is a boundary-defining function satisfying $|d\rho| = 1$ on $b\Omega$. Let $\varphi \in C^1(\overline{\Omega})$ be a real-valued function. Define

$$(f,f)_\varphi = \int_\Omega \langle f, f\rangle e^{-\varphi} dV, \quad f \in \Lambda^{p,q}(U),$$

where $\langle f, f\rangle = \sum_{I,J} |f_{I,J}|^2$ and $\Lambda^{p,q}(U)$ is the space of smooth (p,q)-forms with compact support in U and $\|f\|_\varphi^2 = (f,f)_\varphi$. If

$$f = \sum_{I,J} f_{I,J}\omega^I \wedge \overline{\omega}^J$$

where the sum is over strictly increasing multi-indices of length p and q, respectively, then

$$\bar\partial f = \sum_{I,J} \sum_{j=1}^n \frac{\partial f_{I,J}}{\partial \overline{\omega}^j} \overline{\omega}^j \wedge \omega^I \wedge \overline{\omega}^J + \cdots, \tag{2.1}$$

where $\frac{\partial}{\partial\omega^1}, \cdots, \frac{\partial}{\partial\omega^n}$ are a basis of $T^{1,0}$ that is dual to $\omega^1, \cdots, \omega^n$, and the dots indicate terms in which no $f_{I,J}$ is differentiated; they occur because $\bar\partial\,\omega^i$ and $\bar\partial\,\overline{\omega}^j$ need not be 0. Let $\mathcal{D}^{(p,q)}(U)$ be the space of (p,q)-forms f on U such that

$$f_{I,J} = 0 \quad \text{on } b\Omega \quad \text{when } n \in J. \tag{2.2}$$

Let $\bar\partial^*$ be the L^2-adjoint of $\bar\partial$. For forms $f \in \mathcal{D}^{(p,q)}(U)$ we have

$$\bar\partial^* f = (-1)^{p-1} \sum_{I,K} \sum_{j=1}^n \frac{\partial f_{I,jK}}{\partial\omega^j} \omega^I \wedge \overline{\omega}^K + \cdots, \tag{2.3}$$

where the dots again indicate terms where no derivatives occur in f. If Af denotes the sum in (2.1), then we obtain

$$(2.4)\qquad \|Af\|^2_\varphi = \sum_{I,J}\sum_{j=1}^{n}\left\|\frac{\partial f_{I,J}}{\partial\overline{\omega}^j}\right\|^2_\varphi - \sum_{I,K}\sum_{j,k=1}^{n}\left(\frac{\partial f_{I,jK}}{\partial\overline{\omega}^k}, \frac{\partial f_{I,kK}}{\partial\overline{\omega}^j}\right)_\varphi .$$

Let Bf denote the sum in (2.3). With the notation

$$\delta_j^\varphi\omega := e^\varphi\frac{\partial}{\partial\omega^j}(e^{-\varphi}\omega),$$

we obtain that

$$(2.5)\qquad Bf = (-1)^{p-1}\sum_{I,K}\sum_{j=1}^{n}\delta_j^\varphi f_{I,jK}\omega^I\wedge\overline{\omega}^K + (-1)^{p-1}\sum_{I,K}\sum_{j=1}^{n}\frac{\partial\varphi}{\partial\omega^j}f_{I,jK}\omega^I\wedge\overline{\omega}^K.$$

Since Af and Bf differ from $\overline{\partial}f$ and $\overline{\partial}^*f$ by terms of order zero in f, it follows from (2.4) and (2.5) that

$$\sum_{I,K}\sum_{j,k=1}^{n}(\delta_j^\varphi f_{I,jK},\delta_k^\varphi f_{I,kK})_\varphi - \left(\frac{\partial f_{I,jK}}{\partial\overline{\omega}^k}, \frac{\partial f_{I,kK}}{\partial\overline{\omega}^j}\right)_\varphi + \sum_{I,J}\sum_{j=1}^{n}\left\|\frac{\partial f_{I,J}}{\partial\overline{\omega}^j}\right\|^2_\varphi$$
$$\leq 4\|\overline{\partial}^*f\|^2_\varphi + 2\|\overline{\partial}f\|^2_\varphi + 2\sum_{I,K}\left\|\sum_{j=1}^{n}\frac{\partial\varphi}{\partial\omega^j}f_{I,jK}\right\|^2_\varphi + C\|f\|^2_\varphi,$$

where C is a constant independent of φ. Since the support of f intersects the boundary $b\Omega$, there can be certain boundary integrals. Those that involve the coefficients $f_{I,J}$ for J with $n\in J$ must vanish because of (2.2) or because $\frac{\partial}{\partial \omega^i}, i = 1, \cdots, n-1$, is tangent to $b\Omega$. We obtain

$$(2.6)\quad \int_{U\cap\Omega} \sum_{I,K} \sum_{j,k=1}^{n} \varphi_{jk} f_{I,jK} \overline{f_{I,kK}} e^{-\varphi} dV + \frac{1}{2} \sum_{I,J} \sum_{j=1}^{n} \left\| \frac{\partial f_{I,J}}{\partial \overline{\omega}^j} \right\|_{\varphi}^2$$

$$+ \int_{U\cap b\Omega} \sum_{I,K} \sum_{j,k=1}^{n-1} \rho_{jk} f_{I,jK} \overline{f_{I,kK}} e^{-\varphi} dS$$

$$\leq 4\|\bar{\partial}^* f\|_{\varphi}^2 + 2\|\bar{\partial} f\|_{\varphi}^2 + 2\sum_{I,K} \left\| \sum_{j=1}^{n} \frac{\partial \varphi}{\partial \omega^j} f_{I,jK} \right\|_{\varphi}^2 + C'\|f\|_{\varphi}^2,$$

where C' is a constant independent of φ. Now suppose that $0 \leq \lambda \leq 1$ on $\overline{\Omega}$. Let $\chi(t)$ denote the function $\frac{1}{6}e^t$. Set $\varphi = \chi(\lambda)$. Then

$$\sum_{j,k=1}^{n} \varphi_{jk} t_j \bar{t}_k = \chi'(\lambda) \sum_{j,k=1}^{n} \lambda_{jk} t_j \bar{t}_k + \chi''(\lambda) \left| \sum_{j=1}^{n} \frac{\partial \lambda}{\partial \omega^j} t_j \right|^2 .$$

Since $\chi''(t) \geq 2(\chi'(t))^2$, $\chi'(t) \geq \frac{1}{18}$, it follows from (2.6) that

$$(2.7)\quad \frac{1}{18} \sum_{I,K} \sum_{j,k=1}^{n} \int_{U\cap\Omega} \lambda_{jk} f_{I,jK} \overline{f_{I,kK}} e^{-\varphi} dV \leq 4\|\bar{\partial}^* f\|_{\varphi}^2 + 2\|\bar{\partial} f\|_{\varphi}^2 + C'\|f\|_{\varphi}^2.$$

3. Proof of Theorem

Proof of Theorem. By continuity of the second derivatives of λ, there exists a neighborhood U (dependent on M) of z_0 such that

$$(3.1)\quad \sum_{j,k=1}^{n} \lambda_{jk}(z) t_j \overline{t_k} \geq M|t|^2 \quad , z \in U \cap \overline{\Omega}.$$

Since $\frac{1}{2} \leq e^{-\varphi} \leq 1$, it follows from (2.7) that

$$\frac{M}{36} \int_{U\cap\Omega} |f|^2 dV \leq 4\|\bar{\partial}^* f\|^2 + 2\|\bar{\partial} f\|^2 + C'\|f\|^2.$$

Let $S_\delta := \{z \in X : -\delta < \rho(z) \leq 0\}$. Since $b\Omega$ is compact, we can cover $b\Omega$ by a finite number of such neighborhoods $U_1, \ldots, U_l$ such that $S_\delta \Subset \cup_{\nu=1}^{l} U_{\nu=1}$ for some positive number δ (dependent on M). Thus it follows that

$$(3.2) \qquad M \int_{S_\delta} |f|^2 dV \leq C(\|\bar\partial^* f\|^2 + \|\bar\partial f\|^2 + \|f\|^2)$$

where C is a constant independent of f. Choose $\gamma_\delta \in C_0^\infty(\Omega)$ so that $0 \leq \gamma_\delta \leq 1$ and $\gamma_\delta(z) = 1$ whenever $\rho(z) \leq -\delta$. For a constant a still to be determined, we have the inequality $\|\gamma_\delta f\|^2 \leq a\|\gamma_\delta f\|_1^2 + a^{-1}\|\gamma_\delta f\|_{-1}^2$. By Garding's inequality, there is a constant C_1 depending only on the diameter of the domain Ω such that $\|\gamma_\delta f\|_1^2 \leq C_1(Q(\gamma_\delta f, \gamma_\delta f) + \|\gamma_\delta f\|^2)$. Now $\|\gamma_\delta f\|^2$ can be estimated by

$$\begin{aligned} \|\gamma_\delta f\|_1^2 \leq & 2C_1(\|\gamma_\delta(\bar\partial^* f)\|^2 + \|\gamma_\delta(\bar\partial f)\|^2 + \|\gamma_\delta f\|^2) \\ & + 2C_1(\|[\gamma_\delta, \bar\partial^*]f\|^2 + \|[\gamma_\delta, \bar\partial]f\|^2). \end{aligned}$$

Since the sum of the commutator terms is bounded by $C_2\|f\|^2$ for some constant C_2 dependent on δ, we obtain the inequality

$$(3.3) \qquad \|\gamma_\delta f\|^2 \leq 2aC_1 Q(f,f) + 2aC_1C_2\|f\|^2 + a^{-1}\|\gamma_\delta f\|_{-1}^2.$$

Now choose a so that $2aC_1 < \frac{1}{M}$ and so that $2aC_1C_2 < \frac{1}{2}$. By combining (3.2) and (3.3) we obtain

$$\begin{aligned} M\|f\|^2 &\leq M \int_{S_\delta} |f|^2 dV + M\|\gamma_\delta f\|^2 \\ &\leq C(Q(f,f) + \|f\|^2) + 2aC_1 M Q(f,f) \\ &\quad + 2aC_1C_2M\|f\|^2 + a^{-1}M\|\gamma_\delta f\|_{-1}^2 \\ &\leq (C+1)Q(f,f) + (C + \frac{M}{2})\|f\|^2 + \frac{M}{a}\|\gamma_\delta f\|_{-1}^2, \end{aligned}$$

which gives

$$\|f\|^2 \leq \frac{2(C+1)}{M-2C} Q(f,f) + \frac{2M}{a(M-2C)}\|\gamma_\delta f\|_{-1}^2.$$

Now if we choose M so $\dfrac{2(C+1)}{M-2C} < \varepsilon$ and set

$$\zeta_\varepsilon(z) := \left(\frac{2M}{a(M-2C)}\right)^{\frac{1}{2}} \gamma_\delta(z),$$

then we obtain the compactness estimate $\|f\|^2 \le \varepsilon Q(f,f) + \|\zeta_\varepsilon f\|^2_{-1}$.

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Department of Mathematics
Pusan National University
Pusan 609-735, Korea