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Asset Pricing in the Presence of Taxes: An Empirical Investigation Using the Cox-Ingersoll-Ross Term Structure Model Under Differential Tax Regimes

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Abstract

Relatively little is known about the relationship between taxes and asset prices. Differential tax treatment of assets in the same risk class implies differential pricing. Conversely, the ability of tax-exempt investors to engage in tax arbitrage should drive any pricing differences away. The differential tax treatment of classes of US Treasury securities provides a straightforward setting for the examination of possible tax-effects in asset prices. Using the Cox-Ingersoll-Ross Term Structure Model as our framework, we examine the pricing of US Treasury securities over two distinct tax regimes. Evidence that tax effects are not arbitraged away is presented.

Little is known about the quantitative impact of specific tax policies on asset prices. If tax-effects can be documented to exist in bond prices, it follows that

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there are prescriptions for investor behavior, particularly for tax-exempt investors. If no tax-effects exist, it follows that there are different prescriptions for investor behavior, particularly for taxable investors. Given the changes in the US Tax Code in 1986[1]--some of which have a specific impact upon the taxation of government and corporate bonds--we are provided with an opportunity to answer more persuasively the question of whether tax-effects exist in Treasury security prices. By documenting the quantitative tax-effects, we provide an important understanding of how taxes can affect asset prices.

The term structure of interest rates is the relationship of the required interest rates (yields) on default risk-free pure discount instruments which differ only in their term to maturity. The understanding of these interest rates is integral to the field of finance. Recently, the gross the dollar volume of transactions in the market for default risk-free (US Treasury) debt has approximated \$150 - 200 billion per day--an order of magnitude roughly 20 times that on the NYSE.

Significant advances in term structure modeling have occurred over the past 20 years. These advances have substantially improved the ability of financial economists to draw inferences about factors affecting asset prices from empirical research. The specification of the various term structure models has received much attention in the academic literature. If a given term structure model can be shown to be well-specified, it will likely be of great value to participants in the Treasury markets. In addition, these interest rates are the benchmark rates for pricing most financial assets. Consequently, the importance of the term structure extends well beyond that of the Treasury markets themselves.

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1) Prior to 1987, capital gains on assets held for more than one year were afforded preferential tax treatment under the US Tax Code for individuals. The degree of preference has been variable (most recently a 60% exemption). This differential was removed from The Code in November, 1986, with most of the changes becoming effective on January 1, 1987. See Constantinides and Ingersoll (1984) for a historical summary of changes in the US Tax Code.

The remainder of the paper is organized as follows: In Section I we discuss taxation and asset pricing. In Section II we describe the term structure literature. In Section III we present the Cox-Ingersoll-Ross (1985) Term Structure Model, and the Brown and Dybvig (1986) modifications employed to facilitate empirical testing. In Section IV we develop the reformulation of the model to be tested, and discuss issues related to the methodology. In Section V we describe the data. In Section VI we present our findings on model specification. In Section VII we present the evidence regarding the existence of tax-effects in US Treasury security price data. In Section VIII we test for changes in model pricing errors around the 1986 change in the US Tax Code. Section IX concludes with a summary and suggestions for future research.

I. Taxes and Asset Prices

Taxation of income and capital gains reduces the after-tax wealth of investors. This reduction in after tax wealth would have a well-defined affect on asset prices in the presence of complete markets. In specific, prices would be expected converge to a level which minimized taxation in aggregate.[2] The same would hold if tax rates were equal across all investors. Prices would reflect this equal taxation though a uniform reduction in value proportional to the reduced after-tax cash flows.

Schaefer (1982) provides evidence for the existence of tax clienteles in the United Kingdom Gilt Market. Using a strategy based upon replicating portfolios,

2) See Litzenberger and Rolfo (1984), pages 4-6, for a more detailed discussion on market completeness and tax arbitrage.

he demonstrates that some bonds are dominated (i.e. they are "too expensive") for various tax brackets. The dominance is a function of the tax differential between capital gains and ordinary income. In general, the lower the coupon, the higher the income tax bracket clientele. The reason being that a greater percentage of the returns on such issues is in the form of relatively lower-tax capital gains. He also demonstrates the sensitivity of Gilt prices to changes in the British Tax Code.

Litzenberger and Rolfo (1984) estimate the term structure using a cubic spline methodology. The main focus of their work is the explicit incorporation of a tax function during the estimation procedure. They proceed to fit the data over a variety of tax rates using government bond data from Germany, Japan, the United Kingdom, and the United States. They find that the incorporation of taxes in the estimation procedure reduces the unexplained variation in all cases; significantly so for the UK and the US. Income tax rates implied by their results are shown to be consistent with actual tax rates prevailing for major non-tax-exempt investors in the respective countries.

Some important questions remain be answered. First, given the existence of tax-exempt institutions and the availability of tax-exempt municipal bonds, why should taxable investors set prices for these securities at the margin? Second, asset markets are not complete. However, the US Treasury markets are probably the best approximation of complete markets extant for empirical testing. Over the past 15 years there has been a dramatic increase in volume and in the array of instruments[3)] traded in these markets. Given the recent increases in breaadth and depth, are the US Treasury markets now sufficiently complete to cause any

3) For example the stripping of coupon payments, which are then resold as zero coupon instruments. A cash flow stripped from a premium bond should be no different than a cash flow stripped from a discount bond. Arbitrage arguments suggest that valuation differentials in the presence of the ability to strip coupons should be forced to zero.

tax-induced differences to be driven to zero? Third, if such effects exist, are they in fact only a function of a differential taxation of capital gains as compared to ordinary income? The answers to these questions have important ramifications for the participants in these markets.

II. Term Structure Modeling

Financial economists have devoted considerable effort to the development of mathematical models of the term structure. Early explanations for the existence of term structure have centered around three theories: investor expectations, Fisher (1896), Lutz (1940); liquidity preference, Hicks (1939); and market segmentation, Culbertson (1957), Modigliani and Sutch (1966). Each of these theories appeal to an intuitively plausible aspect of investor behavior.

Over the past two decades there have been significant advances in term structure modeling. As a result, the understanding of factors underlying these interest rates has also advanced. These newer models are built upon the random nature (or stochastic properties) of interest rates. Using the stochastic process as a foundation it is possible to incorporate various factors presumed to underlie the term structure, and to do so in a way which is empirically testable. It follows that the plausibility of any results obtained can be evidenced, in part, by the empirical accuracy of model specification.

Ho & Lee (1986) develop a partial equilibrium term structure model in a discrete time setting. Movements in the term structure through time are modeled according to a binomial lattice approach. The parameters of the model are exogenously derived, as the existing term structure is taken as a given. The model has considerable appeal because it is relatively simple to implement in practice. It is particularly useful in the pricing of interest rate contingent claims, such as interest rate options, callable bonds, putable bonds, and floating rate notes.

Black, Derman, and Toy (1990) modify the work of Ho and Lee. They do so by including a time-varying volatilities for each of the spot rates. Their model also requires that estimates conform with known term structure variables. Model estimation is done recursively.

Heath, Jarrow, & Morton (1992) extend the work of Ho and Lee. The parameters of the model are exogenously derived, as the existing term structure is taken as a given. However, the evolution occurs through forward rates rather than through bond prices. In addition, their specification precludes negative interest rates.

Cox, Ingersoll, and Ross (henceforth CIR) (1985) take a different approach with the development of a continuous-time general equilibrium model. Their model is consistent with rational expectations and utility maximizing behavior. The underlying parameters are endogenously derived in their specification. At the heart of the model is the mean reverting interest rate process, which is known as the "single factor" in the CIR Model. Moreover, the CIR Model provides a relatively concise, empirically testable model of the term structure of interest rates.

Term structure research is too voluminous for all of the important papers in the field to be described here, even in summary form. For additional background, some of the other relatively recent articles in this area are: Brennan and Schwartz (1979), Longstaff (1989), Ramaswamy and Sundaresan (1986), Richard (1978), Schaefer and Schwartz (1984), and Vasicek (1977). See Abken (1990) for a literature review.

III. The CIR Model

The accuracy of the CIR Model, and hence its usefulness in asset pricing theory and practice, was first addressed by Brown and Dybvig (henceforth B&D) (1986). Their approach involves cross sectional estimation of combinations of parameters as originally defined in the model. In general, they find the model to be misspecified, with apparent systematic biases in the residuals relating to the price of the issue. It is conjectured that these biases may be due to some neglected tax-effect, as taxes are not included in the model.

A. The CIR Model.

The Cox-Ingersoll-Ross Model, in its single factor form, is driven by an interest rate process which is:

$$dr = k(\theta - r)dt + \sigma\sqrt{r}\,dz, \tag{1}$$

where: k = the speed of adjustment,

q = the steady state nominal interest rate,

r = the instantaneous nominal interest rate,

$\sigma\sqrt{r}$ = the proportional standard deviation,

dz = a Wiener process.

The price of any pure discount (zero-coupon), default risk-free bond is shown to be:

$$P(r, t) = A(t)*\exp(-B(t)*r), \tag{2}$$

$$A(\tau) = \left[\frac{2\gamma \exp((k+\lambda+\gamma)\tau/2)}{(k+\lambda+\gamma)(\exp(\gamma\tau)+2\gamma} \right]^{2ktheta/\sigma\sigma}, \quad (3)$$

$$B(\tau) = \left[\frac{2(\exp(\gamma\tau)-1)}{(k+\lambda+\gamma)(\exp(\gamma\tau)-1)+2\gamma} \right], \quad (4)$$

where

$$\gamma = [\ (k+\lambda)^2 + 2\sigma^2]^{1/2}, \quad (5)$$

τ = term to maturity,

λ = the risk premium parameter related to term to maturity.

B. B&D Modifications.

In order to facilitate their empirical work, B&D modify the model to conduct an analysis utilizing coupon as well as discount bonds. The true value, V*(t,c,i), of a default risk-free bond is:

$$V^*(t,\ c,\ i) = \Sigma_{i>t}\ c_i P(r,\ \tau_i), \quad (6)$$

where: t = the time of evaluation,

c = the vector of cash flows of the bond,

i = the vector of dates of receipt of the cash flows,

and $P(r,\ \tau)$ is as previously defined.

In essence, this approach allows for the valuation of coupon bonds as a portfolio of pure discount issues, with each coupon payment representing a zero-coupon bond. For true pure discount bonds, such as US Treasury Bills, there is only one cash flow, and equation 6 becomes equivalent to equation 2. They further modify the model to allow for the possibility that the observed price, V(t,c,i), may differ from the true value by adding an error term such that:

$$V(t, c, i) = V^*(t, c, i) + \varepsilon_\tau, \tag{7}$$

and ε_τ is i.i.d. $\sim N(0, \sigma_t^2)$.

To simplify the estimation procedure, they substitute as follows:

$$\phi_1 = [(k + \lambda)^2 + 2\sigma^2]^{1/2}, \tag{8}$$

$$\phi_2 = (k + \lambda + \phi_1)/2, \tag{9}$$

$$\phi_3 = 2k\theta/\sigma^2. \tag{10}$$

Therefore the model estimated is:

$$P(r, \tau) = A(\tau)^* \exp(-B(\tau)^* r), \tag{11}$$

$$A(\tau) = \left[\frac{\phi_1 \exp(\phi_2 \tau)}{\phi_2(\exp(\phi_1 \tau) - 1) + \phi_1} \right]^{\phi_3}, \tag{12}$$

$$B(\tau) = \left[\frac{(\exp(\phi_1 \tau) - 1)}{\phi_2(\exp(\phi_1 \tau) - 1) + \phi_1} \right]. \tag{13}$$

IV. Methodology

A. The Reformulation of the Model

We have generally followed the methodology as set forth in B&D, with the exception that we have exploited the relationships required of the various parameters for mathematical consistency. Consequently, our estimates should be expected to be finer, in the context of the model. Specifically, these relationships are:

$\phi_1 > 0, \quad \forall$ values of k, λ, and σ.

Proof: See appendix A.

$\phi_2 > 0, \quad \forall$ values of k, λ, and σ.

Proof: See appendix B.

$\phi_1 > \phi_2. \quad \forall$ values of k, λ, and σ.

Proof: See appendix C.

Therefore, we have specified these relationships in the test methodology to eliminate the possibility of erroneous estimates such that $\phi_1 > \phi_2$. This has been imposed in the model by redefining:

$$\phi_1 = \phi_2 + \phi_4, \tag{14}$$

where $\phi_4 > 0$.

Accordingly, the reformulated model which we use to estimate the parameters is:

$$V(t, c, i) = V^*(t, c, i) + \varepsilon_\tau, \tag{15}$$

$$V^*(t, c, i) = \Sigma_{i>t}\, c_i P(r, \tau_i), \tag{16}$$

$$P(r, \tau) = A(\tau)^* \exp(-B(\tau)^* r), \tag{17}$$

$$A(\tau) = \left[\frac{(\phi_2 + \phi_4)\exp(\phi_2 \tau)}{\phi_2(\exp(\phi_2 + \phi_4)\tau) - 1) + (\phi_2 + \phi_4)} \right]^{\phi_3}, \tag{18}$$

$$B(\tau) = \left[\frac{(\exp((\phi_2 + \phi_4)\tau) - 1)}{\phi_2(\exp(\phi_2 + \phi_4)\tau) - 1) + (\phi_2 + \phi_4)} \right]. \tag{19}$$

B. Parameter Value Estimation

The parameter estimates for f2, f3, f4, and r are generated utilizing the NLIN (nonlinear regression) procedure available in SAS. "A nonlinear regression model is one for which the first-order conditions for least squares estimation of the parameters are nonlinear functions of the parameters."[4] From an econometric point of view, nonlinear estimation is consistent, although not necessarily the most efficient estimator.

One critical aspect in the implementation of a nonlinear regression analysis is the selection of starting parameter values. While the selection of starting values would appear to be primarily an efficiency issue, these have the potential to affect the estimates if the starting values are far from the true parameter values and the topography of the objective function is "rough".[5] There is little in the way of established procedure for starting parameter value selection methodology. The advice of at least one author is that:

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4) Greene (1990), page 334.

5) For example, if one were to select starting values close to a local minimum which was "far" from the desired global minimum, and the two were separated by an area in which the objective function had values greater than the local minimum.

1. The starting values should be as close to the final (actual) values as possible.
2. Generally trial and error is used to determine what these values should be.[6]

To minimize any deleterious effect related to starting value selection, we have employed the following methodology. First, we selected various months as test periods for parameter estimation. Second, the results obtained during the test periods were further evaluated as dependent variables in the following linear regressions (which were selected by trial and error):

$$\phi_2 = \alpha + \beta_1 \text{Month Nmber} + \beta_2 \text{Invert} + \varepsilon, \tag{20}$$

$$\phi_3 = \alpha + \beta_1 \text{Month Nmber} + \varepsilon, \tag{21}$$

$$\phi_4 = \alpha + \beta_1 \text{Month Nmber} + \beta_2 \text{Invert} + \varepsilon. \tag{22}$$

Month number = 325 to 805, corresponding with 1952:12 to 1992:12. Invert is a dichotomous variable = 1 when the yield curve was inverted, and 0 otherwise. The results of these regressions are shown in Table I. Finally, the coefficient estimates were used to generate the starting values for our 481 month sample period.

In a nonlinear regression, the researcher must specify a point of convergence. This is defined in terms of improvement of squared residuals as the parameter values change for each successive iteration. In this study, convergence criterion was set at:

$$\frac{\text{SSE}_{i-1} - \text{SSE}_i}{\text{SSE}_i + 10^{-6}} < 10^{-4}, \tag{23}$$

6) Greene (1990), page 338.

where i is one iteration. The above convergence criterion is met for every estimation period (month). Summary statistics for parameter estimates are displayed in Table II.

V. The Data

Data on US Treasury securities is available from The Center for Research in Securities Prices at The University of Chicago from December, 1925. To make our results directly comparable to previous work, we have incorporated the entire time frame of the work by B&D--1952:12 to 1983:12--and have added data from 1984:01 to 1992:12. From the aggregate data set, we excluded all callable bonds, tax-exempt bonds, bonds with special tax features (flower bonds), bonds with limited negotiability because of restrictions on bank ownership, bonds with unusual coupon payment cycles (i.e. other than semi-annually or discount issues), and bonds with unusual first coupon payments. The prices of these bonds are: first, transaction prices, if available; second, bid prices, if no transaction prices were available; or third, ask prices; all plus accrued interest as of the observation date. Any observation for which the price is missing is excluded from the analysis. There are 30,953 observations over 481 months in the sample.

VI. Model Specification

In order to draw inferences with respect to tax effects we must first determine that the CIR Model is reasonably well-specified. To assess the specification of the model we examine three sets of results. First, we compare estimates of the instantaneous interest rate with observed rates on short-term T-Bills. Second, we compare long-term rate estimates with a long-term benchmark comprised of an average of the observed rates on the longest available maturities for each observation period. Third, we compare prices predicted by the model with the actual prices observed. For each set of results, we seek to provide evidence on the nature, sign, and size of the prediction errors generated by the model.

A. Instantaneous Interest Rate Estimates

According to the Local Expectations Hypothesis (LEH)[7], all bonds have the same expected rate of return over some arbitrarily short time frame. This is the role of the instantaneous interest rate in the CIR Model. Under the null hypothesis that the model is correctly specified, the implied instantaneous interest rate should not be significantly different from that observed. We fail to reject the model on this basis.

Our test of the above hypothesis consists of a comparison of mean model estimates of the instantaneous interest rate, ERs, and observed rates on a benchmark. The benchmark is the mean yield to maturity prevailing on Treasury Bills with less than 30 days to maturity, Rs. The instantaneous interest rate is estimated for each of the 481 months in the sample period, and these cross sectional estimates are aggregated into 18 different periods for analysis. The results are displayed in Table III. The critical t values are 2.064 (probability $<$ 0.05) and 2.797 (probability $<$ 0.01) for the 24 month periods. In 3 of the 18 periods (16.7%) the difference between the mean estimate and the benchmark rate is significant at the 0.01 level, in 2 of 18 (11.1%) the difference is significant at the 0.05 level, and in 13 of 18 (72.2%) the difference is not significant at standard

7) See Cox-Ingersoll-Ross (1981), page 777.

levels. The model cannot be rejected on the basis of instantaneous interest rate estimates.[8] See Figure 1 for a graph of the mean instantaneous interest rate prediction errors.

B. Implied Long-Term Interest Rate Estimates

The long-term interest rate is the required yield on a pure discount, default risk-free bond as the term tends to infinity. Knowledge of this rate is useful because, in conjunction with knowledge of the instantaneous interest rate, one can characterize the nature of the term structure (normal, flat, inverted), as well as the size of the term premium. Under the null hypothesis that the model is correctly specified, the implied long-term interest rate estimates should not be significantly different from that observed. We reject the model on this basis, but note the shortcomings of the benchmark.

The implied long-term rate is generated from the parameter estimates for each of the 481 months in the sample period as follows:

$$\text{Implied long-term rate, } R_l = ((\phi_2 + \phi_4) - \phi_2)\phi_3, \tag{24}$$

which is consistent with work conducted in B&D. The benchmark is the mean rate on the longest maturity Treasury Bonds and Notes available as of each observation date,[9] Rl. The mean implied long-term rate, ERl, is then compared to the benchmark rate, as done for the instantaneous rate estimates. These

8) It is noteworthy that these mean differences (Estimated Instantaneous Rate - Mean Actual Rate) are positive in all but 3 cases (83.3%). This may have implications for market segmentation which is discussed in part D of this section.

9) For consistency the minimum maturity of securities used to calculate the benchmark rate has been held constant at 5 years, and all bonds with maturities in excess of this have been included in the calculation. Obviously, this can result in a benchmark which varies substantially in duration as the structure of the Treasury maturities changes over time. However, 5 years is the maximum maturity for which there is at least one observation for each month of the sample period.

results are displayed in Table IV. The critical t-values are 2.064 (probability < 0.05), and 2.797 (probability < 0.01) for the 24 month periods. The difference between the implied and actual mean rate is -0.0001, or 1 basis point over the entire 481 month sample period (t = -0.320), and is not significant at standard levels. In 9 of the 18 periods (50.0%) the difference is significant at the 0.01 level, in 3 of 18 (16.7%) the difference is significant at the 0.05 level, and in 6 of 18 (33.3%) the difference is not significant at standard levels. See Figure 2 for a graph of the mean long-term interest rate prediction errors.

Given that the difference is not significant when calculated over the entire sample period, one may question whether some underlying issue may be influencing the shorter period results. The obvious possibility in this case is that of benchmark stationarity (see footnote number 9) and Figure 3. If the term structure were known to be monotonic, it is unlikely that this would be a problem. We could simply take the benchmark and an implied term premium to project the long-term rate as maturity approaches infinity. However, it is well-known that rates can (and often do) begin to decrease at the long end of the term structure. Consequently, the rate on a 30 year bond may be less than that on a 29 year bond, but equal to that on a 20 year bond. This relationship is not consistent over time. Since we are only in possession of observed rates on finite maturities, we are left to arbitrarily assign a long-term benchmark using the data at hand. This is likely to be fine in an overall sense, but not in a specific sense as the benchmark evolves through time.[10] Nonetheless, we must find in favor of rejection on the basis of the data at hand.

10) The maturity of US Treasury debt has changed dramatically during the sample period. The mean maturity of the benchmark in the first sample month is 5.46 years (duration of 1878.00 days), and is over 14 years (duration of 2807.55 days) in the last month. See Figure 2 for a graph of benchmark duration.

C. Implied Prices

A third test of model specification is to compare actual bond prices to those predicted by the model. In doing so, we eliminate the obvious problem of benchmark specification as discussed in part B of this section. The under the null hypothesis that the model is correctly specified, the prices predicted by the model should not be significantly different from those actually observed.

We find price prediction errors for T-Bills and T-Notes which are significant in a statistical sense, while price prediction errors for T-Bonds are not significantly different from zero. We note that the median errors are generally close to the size of the bid/ask spread. We conclude that there is strong evidence of market segmentation in the data, and that T-Bonds are likely more accurately priced because of their greater number of cash flows. Evidence on market segmentation is discussed in part D of this section. Consequently, we reject the CIR Model on the basis of these pricing errors, but assert that model estimates of T-Bonds are sufficiently accurate to permit their use in drawing inference with regard to tax-effects.

For each security used in the estimation procedure, we have data on observed price and predicted price. In order to test the above hypothesis, we have summarized the information in terms of mean error (ME). Mean error is calculated as:

$$ME = \frac{\sum_{i=1}^{n} \text{Market Price}_i - \text{Predicted Price}_i}{n} \tag{25}$$

for each period as indicated. This has been done over the four approximate decade-long sub-periods, as well as for the entire sample period. In addition, the

ME is summarized by issue type.

These results are presented in Table V. Panel A contains data on the mean error for all issues (T-Bonds, T-Notes, and T-Bills combined). The mean error for the entire sample period is -0.088, or $-0.088 per $100 face value. The results for the other sub-periods are similar in sign and size, and are all significantly different from zero at the 0.01 level. This provides strong evidence of mispricing by the model.

D. Market Segmentation

When the errors are evaluated by issue type (Table V, Panels B, C, and D), we see that the cause of the pricing errors rests with T-Bills and T-Notes. This is strongly suggestive of segmentation of the term structure, particularly at the shortest maturities.

Many investors in fixed income securities are subject to institutional limitations on maturity. One example is that of the money-market mutual fund. By definition these funds have average maturities of less than one year, and in most cases substantially less than one year. It is entirely plausible that extreme liquidity preference could result in downward bias on discount factors for the shortest maturities of T-Bills.[11]

We are suggesting that the are two phenomena at work. First, the CIR Model parameter estimates are "best fits" over the entire spectrum of maturities. Almost one-half of the instruments in our sample are T-Bills. Consequently, this exerts a bias on the estimated required yield on the short end of the term structure. Second, given the possiblity of downward pressure on the very short discount factors as a function of extreme liquidity preference, it is unlikely that the very short-term cash flows on securities with longer maturities (i.e. T-Notes and

11) See Simon (1994) for some evidence on segmentation within the T-Bill market.

T-Bonds) are discounted at the same rate. In essence, the cash flow fits the institutional requirement, but the security generating it does not, and does not get priced at the margin by these (extreme liquidity requirement) investors. See Figure 4 for an example of the effects described above.

In summary, the model instantaneous interest rate estimates[12)] are seen as unduly downward biased. As a result, prices are consistently over estimated, and residuals are consistently negative. The result is logically more severe for T-Notes, as a greater percentage of the total return derives from these apparently inappropriately discounted coupons. For T-Bonds, the result is less problematic, and pricing errors are not significant in a statistical sense.[13)]

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12) Recall the results in Table II. Here the estimated short rate is extremely close, and generally slightly in excess of that on a 14 day T-Bill. Benchmark specification is once again an issue. It is likely that as the benchmark maturities increase, particularly beyond the institutional restrictions placed upon permissible maturities, the model estimates will consistently undershoot the actual required yields.

13) Moreover, some caution is warranted with respect to the interpretation of statistical significance, the magnitude of the results, and the ability to reject the validity of the model in a practical sense. The following passage on the meaning of significance is taken from a well-known textbook on probability and statistics. "It is extremely important for the experimenter to distinguish between an observed value of (the test statistic) that is statistically significant and an actual value of the parameter (estimate) that is *significantly different from* the value...specified by the null hypothesis Ho. Although a statistically significant observed value of (the test statistic) provides strong evidence that (the estimate) is not equal to (the value specified by the null hypothesis), it does not necessarily provide strong evidence that the actual value of (the estimate) is significantly different from (the value specified by the null hypothesis). In a given problem, the tail area corresponding to the observed value of (the test statistic) might be very small; and yet the actual value of (the estimate) might be so close to (the value specified by the null hypothesis) that, for practical purposes, the experimenter would not regard (the estimate) as being significantly different from (the value specified by the null hypothesis). The situation just described can arise when the statistic...is based on a very large random sample." (DeGroot (1986), page 496.)

Our sample size is 30,953 observations. In order for the mean price prediction errors not to be significant at the 0.01 level, holding the standard deviation of the estimate constant, the mean error for the overall period would have to be less than ± $0.010 per $100 face value. Similarly, for the periods 1952-9, 1960-9, 1970-9, and 1980-92, the mean errors would have to be less than ± $0.024, ± $0.013, ± $0.014, and ± $0.019, respectively. Since these issues are traded in discrete price increments, this becomes an issue. The customary price increment for T-Bonds and T-Notes is 1/32%, or $0.03125 per $100 face value. In practical terms, this means that the prediction errors would have to be smaller than the bid/ask spread to be "acceptable" in a statistical sense. Of the mean errors presented in Table V, the largest is approximately 4/32%, and the smallest is approximately 1/2 of 1/32%. Thus rejection of the model in a *statistical* sense

E. Nonparametric Residuals Analysis

The t-tests discussed in part D of this section are conducted under the assumption of normality of the residuals. Analysis of the residuals indicates that they are not normal in distribution, exhibiting both skewness and kurtosis. This does not pose problems with respect to the estimation procedure: nonlinear estimation is consistent, but may not be the most efficient estimation procedure in the case of non-normality of the residuals. However, this may cast doubt upon the reliability of the statistical inferences derived from parametric tests.

To address this concern, we conduct a nonparametric test of the residuals. The test used is the Wilcoxon Signed Rank Test, and the results are displayed in Table VI. The pricing error for the T-Bonds is not significantly different from zero in a statistical sense. The pricing errors for the remainder of the categorical data are statistically significant. The values of the mean errors are the same as those displayed in Table V, and we also include the median errors for comparison. It is noteworthy that the median errors are all negative in sign, consistent with conjecture that segmentation is present in the term structure. We conclude, however, that T-Bond pricing is sufficiently accurate to permit testing for tax-effects in the data.

VII. Tax-Effects

In this section we turn our attention to possible tax-effects in US Treasury security prices. According to the results presented in the previous section, the accuracy of the CIR Model specification could not be rejected for the pricing of

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must also be viewed in the context of the practical value of the pricing errors.

T-Bonds. As these each of these securities (on average) represents a portfolio of a relatively greater number of cash flows, perhaps this is not surprising. Moreover, T-Bonds are likely to be the most sensitive to tax effects. Therefore, we restrict our analysis for tax effects to these issues, and in so doing, we reduce the probability that the tests for tax-effects will be a function of model specification error.

We provide evidence related to the basic question of whether we able to detect tax-effects related to the asymmetrical tax treatment of discount and premium issues. To do so, we consider the size and significance of pricing errors, and correspondence with known features of the Tax Code. The evidence presented in this and the following section is uniformly consistent with the existence of tax-effects in the data.

A. The Rationale For Tax Effects in Bond Prices

B&D find a systematic bias in the pricing of US Treasury issues specifically related to the nature (i.e. whether the bonds were premium (price > 100) or discount[14] (price < 100)) of the issue. They also note that this bias may be tax-related, as discount and premium issues are treated in an asymmetrical fashion with respect to individual income taxes.

Discount bonds are subject to two related differential tax treatments. First, these bonds generally have their discount realized when the bond is sold or matures, causing the capital gain on the investment to be deferred for tax purposes. Even if the aggregate tax liability is the same as if it were realized year by year, this "deferral benefit" is valuable to the taxable investor simply because of the time value of money. Second, under the Tax Code in existence

14) The reference to discount bonds in this section pertains to bonds which were originally issued at or near par value. Original issue discount/pure discount issues are not included.

from 1979 - 1986,[15] the gain was treated as a long-term capital gain on assets held for more than one year, and was taxed at a lower rate than ordinary income. Given the implicit tax advantages of the discount issues, one would expect their prices to exceed (tax neutral) model predictions.

Premium bonds are subject to a single differential treatment. These bonds may have their premium realized when the bond is sold or matures, or it may be amortized over the life of the issue. For taxable investors it is desirable to amortize the premium and realize the loss more quickly because of the time value of money. Premium bonds are at a relative disadvantage for non-tax-exempt investors. Therefore, one would expect that the tax disadvantage of premium bonds, relative to discount bonds, should result in their prices falling short of model predictions.

B. Residuals Analysis: Discount and Premium Issues

In this section, we test our first hypothesis which is summarized in terms of mean error (ME). Mean error is calculated as:

$$ME = \frac{\sum_{i=1}^{n} \text{Market Price}_i - \text{Predicted Price}_i}{n} \tag{25}$$

for each period as indicated. The hypothesis is that of tax-effects related to the asymmetrical tax treatment of discount and premium issues:

H_{01} : $ME = 0, \quad \forall$ bonds,

15) This is the capital gains tax preference as described in Footnote 1.

H_{a1} : ME $\neq$ 0, $\forall$ bonds.

In words, the null hypothesis is that the mean error equals zero for both discount and premium issues, while the alternative states that this will not be the case (i.e. it will be a function of taxes). We reject the null hypothesis in favor of the alternative.

Table VII contains information on the prices implied by the CIR Model for discount and premium issues compared with the actual prices. If there is a tax-induced bias present, one would expect discount issues to exhibit a positive pricing error, and premium issues to exhibit a negative pricing error. The mean error is \$0.245 (median = \$0.095) for discount issues, and \$-0.146 (median = \$-0.116) for premium issues. Both of these are significantly different from zero at the 0.01 level. Moreover, these errors are substantially in excess of the usual bid/ask spread. On this basis, we reject the null hypothesis in favor of the alternative of tax-effects in T-Bond prices. See Figures 5 and 6 for graphs of the prediction errors for discount and premium T-Bonds, respectively.

C. Effect Sensitivity to Size of Premium/Discount

In this section we seek to provide corroborating documentation that the pricing differentials on discount and premium T-Bonds are related to taxes, rather than a function of chance. To do so, we note that the value of the tax differential should be directly related to the size of the discount/premium. Therefore, we replicate the above analysis restricting the sample to issues which have sizable discounts/premiums. Specifically, the sample is restricted to T-Bonds which have prices at least five points above or below par. The hypothesis put forth in the previous section remains the same, and the results are shown in Table VIII.

Panel A shows the results for the discount T-Bonds which have prices less than 95. Under the alternative hypothesis that taxes are driving these differentials, one would expect the size of the error the increase, reflecting the greater degree of tax preference inherent in these "deep" discount issues relative to all discount issues. The mean "deep" discount error is $0.459 per $100 face value, and is significantly different from zero. Moreover, the size of the error has increased substantially from the $0.245 for all discount T-Bonds. The size and sign of the mean errors correspond precisely with the alternative tax-effect hypothesis.

Panel B shows the results for premium T-Bonds with prices greater than 105. Under the alternative hypothesis that taxes are driving these differentials, one would expect the size of the mean premium error to increase, reflecting the even greater asymmetry of tax treatment. The mean "deep" premium error is $-0.306 per $100 face value, and is significantly different from zero. The mean error for all premium T-Bonds was $-0.146. Again, the size and sign of the mean errors correspond precisely with the alternative tax-effect hypothesis. On the basis of the above results, we reject the null hypothesis in favor of the alternative--that of the existence of tax-effects in T-Bond prices.

VIII. Differential Tax Regimes

In this section we address the question of whether we can attribute any change in the size of the pricing errors to the 1986 change in the US Tax Code. The changes effected have a direct bearing on the tax treatment of these securities. The change in the Tax Code in 1986 removed the capital gains tax advantage[16] for discount issues. Given this change in tax treatment, we are provided with an

16) See Footnote number 1.

opportunity to answer more persuasively the question of whether tax-effects exist in US Treasury security prices.

It is reasonable to expect that if taxes are driving the results, there should be a noticeable difference between pre-change and post-change periods. Pre-change and post-change periods have been defined as 1979:01-1986:12 and 1987:01-1992:12, respectively. In effect, we are attempting to test whether the biases detected in Section VII are sensitive to substantive changes in The Code. If so, the obvious implication is that these are a function of taxes. If not, then the implication is that the biases are coincidental to, and not a function of taxes.

As a first step, we test the null hypothesis that the mean error for T-Bonds is zero by tax regime and issue type. These results are shown in Table IX. In all periods the errors are significantly different from zero, and are consistent with results in preceding sections. Tax effects seem to persist even in the absence of a capital gains tax differential. This implies that the "deferral benefit" discussed in the previous section may be significant.

A. Nonparametric Tests for Mean Equality

With the implication that tax-effects are extant over both regimes, the basic question to be answered is whether there is a demonstrable change in the tax-effects compared across pre-change and post-change periods. The hypothesis is that related to capital gains treatment:

$$H_{02} : ME_{pre\text{-}change} = ME_{post\text{-}change}, \quad \forall \text{ bonds.}$$

$$H_{a2} : ME_{pre\text{-}change} \neq ME_{post\text{-}change}, \quad \forall \text{ bonds.}$$

In words, the null hypothesis is that the mean residuals will be the same over pre and post-change periods, while the alternative states that they will not be equal because of the change in The Code. We reject the null in favor of the alternative.

The results in Table X show that residuals distribution is significantly different across the tax regime shift in all cases. The sign and degree of the mean errors for the T-Bonds support the alternative hypothesis exactly: the degree of the tax-effect is sensitive to the change in the tax regime.

Moreover, the change is also economically significant, with the size of the mean error change ranging from as little as \$0.093 per \$100 face value to as much as \$0.60 per \$100 face value, depending upon issue class. Given the results shown in Tables IX and X, we conclude that definitive tax-effects exist in the prices of T-Bonds during both tax regimes, and that the size of these effects are directly affected by the 1986 change in the Tax Code.

IX. Summary

The ramifications of taxation on asset prices is an issue of importance for all investors. Perhaps the most straightforward manner in which to evaluate tax-effects is through the pricing of default risk-free securities. Numerous models of the term structure of interest rates have been proposed to price these securities. Relatively recent advances in these models enhance the plausibility of any tax-effect results derived therefrom.

We have used nonlinear regression to estimate the parameters of the CIR model. These parameters are then used to generate a variety of data pertaining to interest rates and bond prices. The interest rate estimates, particularly those of the instantaneous interest rate, are generally consistent with the data. In cases where statistically significant deviations are observed, it has been emphasized that the choice of bench mark is critical, and may be the source of some of these differences. What appears to be most interesting is the consistency of the sign of the differences for the instantaneous interest rate. This has both theoretical and practical implications, and is worthy of further research effort.

The prices implied by the CIR Model are generally found to be significantly different from those observed in a statistical sense. There are suggestions of market segmentation which may particularly affect model valuation of T-Bills and T-Notes. However, model pricing of T-Bonds is not significantly in error. T-Bond residuals are deemed sufficiently accurate so as to facilitate the unbiased detection of tax-effects.

The primary conclusion of this paper is that tax-effects do exert biases in the pricing of US Treasury Bonds, and that these are significant in both a statistical and a practical sense. Discount issues are consistently overpriced while premium issues are consistently underpriced during the sample period. This is indicative of a tax-effect related to the asymmetrical treatment of discount and premium issues. The secondary conclusion of this paper is that the size of these pricing errors is significantly smaller following the change in tax regimes at the end of 1986. These results have important practical implications, and are indicative of a need for further theoretical work incorporating taxes into term structure models.

Table I Regression Estimates of Parameter Starting Values

Month Number = 325 to 805; 1952:12 = 325, 1992:12 = 805.
Invert = 0 or 1; 1 = Yield curve inverted, 0 = Otherwise.
Standard errors in parentheses.
Observations = 27.

$$\phi_2 = \underset{(0.3584)}{1.5365} - \underset{(0.0007)}{0.0013}\text{ Month Number} + \underset{(0.2089)}{0.5083}\text{ Invert} + \varepsilon$$

$$\phi_3 = \underset{(0.1784)}{-0.5119} + \underset{(0.0003)}{0.0024}\text{ Month Number} + \varepsilon$$

$$\phi_4 = \underset{(0.0210)}{0.0730} + \underset{(0.0000)}{0.0000}\text{ Month Number} + \underset{(0.0123)}{0.0200}\text{ Invert} + \varepsilon$$

Table II Parameter Estimate Data

Observations = 481.

Panel A: Summary Statistics

	Mean	σ	Max.	Min.
ϕ_2	0.0906	0.0384	0.3396	0.0282
ϕ_2	0.9018	0.6065	7.1511	0.0227
ϕ_2	0.8533	0.4767	2.4142	0.0683

Panel B: Correlation Matrix

	ϕ_2	ϕ_2	ϕ_2
ϕ_2	1.0000		
ϕ_2	0.0154	1.0000	
ϕ_2	-0.5269	-0.0857	1.0000

Table III Implied Instantaneous Interest Rate Comparison

$H_0: \ ER_s - R_s = 0$

Panel A: Overall Sample Period and Approximate Decade Sub-Periods:

Time Frame	ER_s(mean)	R_s(mean)	$ER_s - R_s$	$\sigma_{estimate}$	t-value
1952:12 - 1992:12	.0.0541	0.0530	0.0012	0.0071	3.602**
1952:12 - 1959:12	0.0193	0.0191	0.0002	0.0036	0.506
1960:01 - 1969:12	0.0382	0.0376	0.0007	0.0040	1.852
1970:01 - 1979:12	0.0635	0.0619	0.0016	0.0075	2.333*
1980:01 - 1992:12	0.0782	0.0764	0.0018	0.0097	2.263*

Panel B: Two-Year Interval Sub-Periods:

Time Frame	ER_s(mean)	R_s(mean)	$ER_s - R_s$	$\sigma_{estimate}$	t-value
1967:01 - 1968:12	0.0468	0.0460	0.0008	0.0042	0.947
1969:01 - 1970:12	0.0642	0.0624	0.0018	0.0071	1.272
1971:01 - 1972:12	0.0410	0.0399	0.0011	0.0046	1.180
1973:01 - 1974:12	0.0754	0.0753	0.0001	0.0081	0.059
1975:01 - 1976:12	0.0536	0.0513	0.0024	0.0040	2.870**
1977:01 - 1978:12	0.0612	0.0626	-.0015	0.0067	-1.070
1979:01 - 1980:12	0.1103	0.1049	0.0054	0.0148	1.788
1981:01 - 1982:12	0.1227	0.1205	0.0021	0.0117	0.895
1983:01 - 1984:12	0.0861	0.0894	-.0033	0.0085	-1.873
1985:01 - 1986:12	0.0649	0.0628	0.0021	0.0056	1.870
1987:01 - 1988:12	0.0632	0.0533	0.0099	0.0077	6.294**
1989:01 - 1990:12	0.0757	0.0745	0.0012	0.0084	0.676
1991:01 - 1992:12	0.0407	0.0415	-.0009	0.0037	-1.169

* Difference significant at the 0.05 level.

** Difference significant at the 0.01 level.

Table IV Implied Long-Term Interest Rate Comparison

$H_0: ER_1 - R_1 = 0$

Panel A: Overall Sample Period and Approximate Decade Sub-Periods:

Time Frame	ER_1(mean)	R_1(mean)	$ER_1 - R_1$	$\sigma_{estimate}$	t-value
1952:12 - 1992:12	0.0677	0.0678	-.0001	0.0059	-0.320
1952:12 - 1959:12	0.0305	0.0309	-.0005	0.0027	-1.582
1960:01 - 1969:12	0.0457	0.0466	-.0008	0.0046	-2.025*
1970:01 - 1979:12	0.0735	0.0740	-.0006	0.0063	-0.959
1980:01 - 1992:12	0.1004	0.0994	0.0011	0.0073	1.818

Panel B: Two-Year Interval Sub-Periods:

Time Frame	ER_1(mean)	R_1(mean)	$ER_1 - R_1$	$\sigma_{estimate}$	t-value
1967:01 - 1968:12	0.0533	0.0541	-.0008	0.0019	-2.086*
1969:01 - 1970:12	0.0655	0.0706	-.0052	0.0081	-3.120**
1971:01 - 1972:12	0.0654	0.0613	0.0041	0.0037	5.509**
1973:01 - 1974:12	0.0686	0.0727	-.0041	0.0065	-3.078**
1975:01 - 1976:12	0.0768	0.0749	0.0019	0.0018	5.141**
1977:01 - 1978:12	0.0786	0.0781	0.0005	0.0074	0.325
1979:01 - 1980:12	0.0987	0.1041	-.0054	0.0038	-6.893**
1981:01 - 1982:12	0.1239	0.1327	-.0088	0.0090	-4.811**
1983:01 - 1984:12	0.1190	0.1179	0.0011	0.0026	2.085*
1985:01 - 1986:12	0.0956	0.0916	0.0040	0.0027	7.293**
1987:01 - 1988:12	0.0895	0.0860	0.0034	0.0028	5.969**
1989:01 - 1990:12	0.0847	0.0855	-.0008	0.0021	-1.913
1991:01 - 1992:12	0.0850	0.0750	0.0101	0.0054	9.178**

* Difference significant at the 0.05 level.

** Difference significant at the 0.01 level.

Table V Implied Price Comparison

H_0 : ME = 0

Mean Error (ME) = $\sum_{1}^{n}$(Market $Price_i$ - Predicted $Price_i$)/n

Time Frame	ME	σ_{ME}	Obs.	t-value
Panel A: All Issues:				
1952:12 - 1992:12	-0.088	0.710	30953	-21.78**
1952:12 - 1959:12	-0.135	0.425	2152	-14.73**
1960:01 - 1969:12	-0.044	0.376	5610	- 8.79**
1970:01 - 1979:12	-0.086	0.489	7852	-15.61**
1980:01 - 1992:12	-0.098	0.904	15339	-13.46**
Panel B: U.S. Treasury Bonds:				
1952:12 - 1992:12	0.018	1.035	1757	0.72
Panel C: U.S. Treasury Notes:				
1952:12 - 1992:12	-0.129	0.957	14139	-16.06**
Panel D: U.S. Treasury Bills:				
1952:12 - 1992:12	-0.061	0.220	15057	-34.30**

* Difference significant at 0.05 level.

** Difference significant at 0.01 level.

Table VI Nonparametric Residuals Analysis

H_0 : ME = 0

Mean Error (ME) = $\sum_{1}^{n}$(Market $Price_i$ - Predicted $Price_i$)/n

Z_{WSR} = Test statistic generated by the Wilcoxon Signed Rank Test.
p_{WSR} = Probability associated with H_0 ($P(H_0)$: ME = 0).

Time Frame	ME	Median Error	Z_{WSR}	p_{WSR}
		Panel A: All Issues:		
1952:12 - 1992:12	-0.088	-0.021	-31.75	<.0001
1952:12 - 1959:12	-0.135	-0.013	-14.68	<.0001
1960:01 - 1969:12	-0.044	-0.031	-28.09	<.0001
1970:01 - 1979:12	-0.086	-0.012	-14.17	<.0001
1980:01 - 1992:12	-0.098	-0.022	-17.12	<.0001
		Panel B: U.S. Treasury Bonds:		
1952:12 - 1992:12	0.018	-0.045	- 1.62	0.1057
		Panel C: U.S. Treasury Notes:		
1952:12 - 1992:12	-0.129	-0.050	-19.58	<.0001
		Panel D: U.S. Treasury Bills:		
1952:12 - 1992:12	-0.061	-0.014	-28.54	<.0001

Table VII Nonparametric Residuals Analysis: Discount and Premium U.S. Treasury Bonds

$$H_0 : ME = 0$$

$$\text{Mean Error (ME)} = \sum_{1}^{n}(\text{Market Price}_i - \text{Predicted Price}_i)/n$$

Z_{WSR} = Test statistic generated by the Wilcoxon Signed Rank Test.
p_{WSR} = Probability associated with H_0 $(P(H_0) : ME = 0)$.

Time Frame	ME	Median Error	Z_{WSR}	p_{WSR}
Panel A: Discount Issues With Price < 100				
1952:12 - 1992:12	0.245	0.095	4.37	<.0001
Panel B: Premium Issues With Price > 100				
1952:12 - 1992:12	-0.146	-0.116	- 6.27	<.0001

Table VIII Nonparametric Residuals Analysis:
Deep Discount and Premium U.S. Treasury Bonds

$$H_0 : ME = 0$$

$$\text{Mean Error } (ME) = \sum_{1}^{n}(\text{Market Price}_i - \text{Predicted Price}_i)/n$$

Z_{WSR} = Test statistic generated by the Wilcoxon Signed Rank Test.
p_{WSR} = Probability associated with H_0 $(P(H_0) : ME = 0)$.

Time Frame	ME	Median Error	Z_{WSR}	p_{WSR}
Panel A: Discount Issues With Price < 95:				
1952:12 - 1992:12	0.459	0.291	5.61	<.0001
Panel B: Premium Issues With Price > 105:				
1952:12 - 1992:12	-0.306	-0.287	-29.31	<.0001

Table IX Nonparametric Residuals Analysis: Discount and Premium U.S. Treasury Bonds Over Differential Tax Regimes

$$H_0 : ME = 0$$

$$\text{Mean Error (ME)} = \sum_{1}^{n}(\text{Market Price}_i - \text{Predicted Price}_i)/n$$

Z_{WSR} = Test statistic generated by the Wilcoxon Signed Rank Test.
p_{WSR} = Probability associated with H_0 $(P(H_0) : ME = 0)$.

Time Frame	ME	Median Error	Z_{WSR}	p_{WSR}
Panel A: Discount Issues With Price < 100:				
1979:01 - 1986:12	0.624	0.507	5.65	<.0001
1987:01 - 1992:12	0.313	0.177	5.55	<.0001
Panel B: Discount Issues With Price < 95:				
1979:01 - 1986:12	0.754	0.755	5.05	<.0001
1987:01 - 1992:12	0.154	0.122	2.69	0.0062
Panel C: Premium Issues With Price > 100:				
1979:01 - 1986:12	-0.244	-0.162	-2.09	0.0355
1987:01 - 1992:12	-0.151	-0.149	-5.60	<.0001
Panel D: Premium Issues With Price > 105:				
1979:01 - 1986:12	-0.647	-1.020	- 3.26	0.0007
1987:01 - 1992:12	-0.294	-0.287	- 9.39	<.0001

Table X Nonparametric Tests For Equality of Residual Distribution Across Tax Regimes

H_0: Residual Distribution is Equal Across Periods
1979:01 - 1986:12 and 1987:01 - 1992:12

Mean Error (ME) = $\sum_{1}^{n}$(Market $Price_i$ - Predicted $Price_i$)/n

$p\text{-value}_{KS}$ = Probability value associated with H_0 $(P(H_0): F(x) = G(x))$ according to the Kolmogorov-Smirnov Test.

	$ME_{prechange}$	$ME_{postchange}$	$p\text{-value}_{KS}$
Panel A: Discount Issues			
T-Bond, Price < 100	0.624	0.313	0.0004
T-Bond, Price < 95	0.754	0.154	<.0001
Panel B: Premium Issues			
T-Bond, Price > 100	-0.244	-0.151	<.0001
T-Bond, Price > 105	-0.647	-0.294	<.0001

Appendix A

Proof that $\phi_1 > 0$:

$$\phi_1 = [(k+\lambda)^2 + 2\sigma^2]^{0.5} \quad \forall\, k \neq -\lambda, \quad (k+\lambda)^2 > 0,$$

otherwise $(k+\lambda)^2 = 0$.

Since the interest rate is driven by a stochastic process,

$\sigma^2 > 0, \Rightarrow 2\sigma^2 > 0.$

Therefore, $\phi_1 = [(k+\lambda)^2 + 2\sigma^2]^{0.5} > 0.$

Q.E.D.

Appendix B

Proof that $\phi_2 > 0$:

$$\begin{aligned}
\phi_2 &= (k + \lambda + \phi_1)/2 > 0 \\
&= (k + \lambda)/2 + \phi_1/2 > 0 \\
&= (k + \lambda)/2 + [(k+\lambda)^2 + 2\sigma^2]^{0.5}/2 > 0 \\
&= [(k+\lambda)^2 + 2\sigma^2]^{0.5} > [-(k + \lambda)] \\
&= [(k+\lambda)^2 + 2\sigma^2] > [-(k + \lambda)]^2
\end{aligned}$$

Since $(k+\lambda)^2 = [-(k+\lambda)]^2$, and $2\sigma^2 > 0$,

therefore, $\Rightarrow \phi_2 = (k + \lambda + \phi_1)/2 > 0.$

Q.E.D

Appendix C

Proof that $\phi_1 - \phi_2 > 0$:

$$\phi_1 = [\ (k+\lambda)^2 + 2\sigma^2]^{0.5}, \text{ and } \phi_2 = (k + \lambda + \phi_1)/2.$$

So if $[\ (k+\lambda)^2 + 2\sigma^2]^{0.5} - (k + \lambda + \phi_1)/2 > 0,$

then $[\ (k+\lambda)^2 + 2\sigma^2]^{0.5} - (k + \lambda)/2 + \phi_1/2 > 0$

$$[\ (k+\lambda)^2 + 2\sigma^2]^{0.5}/2 - (k + \lambda)/2 > 0$$

$$[\ (k+\lambda)^2 + 2\sigma^2]^{0.5} - (k + \lambda) > 0$$

$$[\ (k+\lambda)^2 + 2\sigma^2] > (k + \lambda)^2.$$

Since $(k+\lambda)^2 = (k + \lambda)^2$, and $2\sigma^2 > 0$,

therefore, $\Rightarrow \phi_1 - \phi_2 > 0$.

Q.E.D

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