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The Effect of Stochastic Taxes on Asset Prices

Chang-Soo Kim*

Abstract

This paper develops an equilibrium asset pricing model with taxation in the economy. The expected excess rate of return on a risky asset is shown to be an increasing function of the covariance of asset return with aggregate consumption rate changes and the covariance of asset return with the tax rates as well. Thus, the expected execss rate of return can be decomposed as the consumption risk premium and the tax premium. The capital asset pricing model derived in the absence of taxes is shown to understate the expected excess rate of return and to have a misspecification error in the economy with taxation.

Ⅰ. Introduction

An important goal of financial research for the last decades has been to generalize the insights of the simple one-period capital asset pricing model (CAPM) to a more realistic and multiperiod setting. For instance, Merton (1969, 1973) derives the equilibrium relations among expected returns in an intertemporal asset pricing model where investors are compensated not only for taking on systematic market risk, but also for bearing the risk of unfavorable shifts in investment opportunity set. As one of the important state variables that characterize the investment opportunities, this paper considers stochastic tax rates.

In recent years, there have been many tax code changes. For example, the Economic Recovery Tax Act of 1981 led to the largest postwar decline in effective tax rates on

*Assistant professor, Department of Business Administration, Yonsei University

capital. Accompanying this tax cut is the economic recovery, which began in November 1982. During this recovery there have been relatively large increases in business fixed investment, a stock market boom, and a large rise in both the ex post and ex ante real interest rate. On July 18, 1984, a sweeping package of new U.S. tax regulations was signed into law. One purpose of this legislation was to increase the risk and reduce the profitability of dividend-related trading by incorporated investors. Furthermore, the Tax Reform Act of 1986 included the most extensive changes in the U.S. Tax law since the dramatic increase in corporate and personal tax rates during World War II. The maximum tax rate for corporate income dropped from 46 percent to 34 percent and that for individual income from 50 percent to 28 percent. Interest on consumer credit was phased out, and capital gains income became taxable as ordinary income. Since these tax code changes are ex ante uncertain and yet have a profound impact on the economy, it is important to investigate the linkage between the tax rate changes and the investment opportunities.

In the extant literature, most of the work dealing with the effects of taxes assumed that agents have perfect foresight regarding the path of tax rates (e.g. Becker (1985) and Goulder and Summer (1987)). Little effort seems to have been given to examining the effects of tax rate changes when taxes are explicitly depicted as following a particular stochastic process.[1] In some cases, a model which assumes deterministic tax rates produces a conflicting result from the model which employs uncertain tax rates. For example, Mauer, Barnea, and Kim (1991) demonstrate that the issuance of a callable bond is a negative sum game when tax rates and interest rates positively covary. The result is opposite to the results obtained in other studies which have offered tax-based rationales for the widespread practice of attaching call provisions to corporate bonds assuming deterministic tax rates. Thus, the objective of this paper is to introduce stochastic tax rates in order to derive an intertemporal equilibrium asset pricing model under more realistic economic setting. The model is an extension of the consumption based asset pricing model developed by Breeden (1979) in the absence of taxes. The analysis shows that there will be a positive tax premium for a risky asset as long as the return on the asset and the tax rate is positively correlated. Therefore, the omission of the tax premium will understate the expected excess rate of return.

Empirically, criticizing the studies wherein the market is the sole risk factor, several recent papers consider multiple risk factors that can account for possible variations in the investment opportunity set that are ignored by the conditional CAPM.[2] Since the general equilibrium effects of taxes on the asset prices seem to be well understood, incorporating taxes can contribute to empirical studies examining the predictability of asset returns.

The paper is organized as follows. Section II presents assumptions and the economic environment. Section III derives the intertemporal capital asset pricing model under taxation. Section IV examines empirical implications of the model and Section V concludes the paper.

Ⅱ. Set-up

This section develops a model of asset prices in a simple economic setting. The following assumptions characterize the economy.

Assumption 1: There are n+1 securities and the (n+1)th security is riskless.

Assumption 2: The price processes of n+1 securities are governed by a system of stochastic differential equations of the form,

$$\begin{aligned} \frac{dP_i(t)}{P_i(t)} &= R_i(X,t)dt + \sigma_i(X,t)dz_i(t), \qquad \text{for } i = 1, 2, \ldots, n, \text{ and} \\ &= r(X,t)dt \qquad\qquad\qquad\qquad\quad \text{for } i = n+1, \end{aligned} \tag{1}$$

where $P_i(t)$ is the price of security i at time t, X is an m-dimensional vector of state variables whose movement will be described shortly, $R_i(X,_t)$ is the instantaneous conditional expected percentage change in price of asset i per unit of time, $\sigma_i(X,_t)$ is the instantaneous conditional standard deviation of the percentage change in price of asset i, and $dz_i(t)$ is the increment of a standard Gauss-Wiener process.

Assumption 3: The movement of the m-dimensional vector of state variables, X, is determined by a system of stochastic differential equations of the form,

$$dX = \mu(X,t)dt + G(X,t)de(t). \tag{2}$$

where $\mu(X,t)$ is an m-dimensional vector of conditional mean, and $G(X,t)$ is an m x m diagonal matrix whose jth diagonal element is the instantaneous standard deviation of the change in jth state variable, and $de(t)$ is the increment of a Gauss-Wiener process. The state variable process represents the change in the economic environment which has an influence on the investment opportunities.

Assumption 4: The tax rate follows a stochastic AR(1) process of the form,

$$dT(t) = -aT(t)dt + T_0 dt + T_1 db(t), \tag{3}$$

where $T(t)$ is a tax rate at time t, $db(t)$ is the increment of a Gauss-Wiener process, and a, T_0 and T_1 are constant.

Assumption 5: The taxable income of an investor k is determined by the sum of capital gains or losses and non-capital earnings, and the taxable income process is governed by a stochastic differential equation of the form,

$$dV^k(t) = \sum_{i=1}^{n+1} q_i^k(t) W^k(t) \frac{dP_i(t)}{P_i(t)} + Y^k(t)dt, \tag{4}$$

where $V^k(t)$ is a taxable income of investor k at time t, $q_i^k(t)$ is a fraction of wealth of investor k invested in security i at time t, $W^k(t)$ is the after-tax wealth of investor k at time t, and $Y^k(t)$ is the rate of non-capital earnings (presumably wages) of investor k at time t. In equation (4), the after-tax wealth of investor k at time t, $W^k(t)$, is equal to

$$W^{\iota}(t) = V^k(t) - T(t)V^k(t) - C^k(t), \tag{5}$$

where $C^k(t)$ is the cumulative consumption of investor k to time t. That is, the after-tax wealth of investor k is determined by the taxable income less tax payment and consumption up to time t. Then, by Ito's Lemma, the process of after-tax wealth can be derived as

$$dW^k(t)=\Big[-c^k(t)+\{1-T(t)\}\Big(\sum_{i=1}^{n+1}q_i^k(t)W^k(t)R_i(X,t)+Y^k(t)\Big)$$
$$+V^k(t)\{aT(t)-T_0\}-T_1\sum_{i=1}^{n+1}q_i^k(t)W^k(t)\sigma_i(X,t)\rho(T,R_i)\Big]\,dt \tag{6}$$
$$+\{1-T(t)\}\Big[\sum_{i=1}^{n+1}q_i^k(t)W^k(t)\sigma_i(X,t)\Big]\,dz_i(t)-V^k(t)T_1db(t),$$

where $c^k(t)$ is the consumption rate of investor k at time t and $\rho(T, R_i)$ is the correlation coefficient between tax rates and asset i.

Assumption 6: There are a fixed number of investors. All investors agree that security prices, state variables, and tax rates are as described. Investor k seeks to maximize an objective function of the form,

$$\max_{c^k,q^k} E_t\int_t^{D^k} U^k(c^k, s)ds+B^k[W^k(D^k), X(D^k), T(D^k), D^k], \tag{7}$$

where Et is an expectation operator conditional on current endowment and the state of the economy, $U^k(\cdot)$ is a von Newmann-Morgenstern utility function of investor k which is increasing, strictly concave, and twice differentiable, $B^k(\cdot)$ is a bequest function of investor k, and D^k is the time of death for investor k.

Ⅲ. Intertemporal Asset Pricing under Stochastic Taxes

Investor k attempts to maximize his expected lifetime utility by making optimal decisions on consumption and investment intertemporally. The expected utility maximization can be specified as the following stochastic control problem.[3]

$$J^k(W^k, X, T, t)=\max_{c^k,q^k} E_t\int_t^{D^k} U^k(c^k,s)ds+B^k[W^k(D^k), X(D^k), T(D^k), D^k], \tag{8}$$

where $J^k(W^k,X,T,t)$ is the maximum expected utility of investor k from time t to D^k.

For the maximization problem, the Hamilton-Jacobi-Bellman equation is

3) For a detailed discussion of the optimal stochastic control, see Malliaris and Brock (1988).

$$
\begin{aligned}
0=&\max_{c^k,q^k} U^k(c^k,s)+J^k_t \\
&+J^k_m[-c^k+(1-T)\{W^k r+W^k(q^k)'(R-r)+Y^k\}+V^k(aT-T_0)-W^k(q^k)'\Psi_{RT}\}] \\
&+\mu' J^k_X+J^k_T(-aT+T_0) \\
&+\frac{1}{2}J^k_{WW}[(1-T)^2(W^k)^2(q^k)'\Psi q^k+(V^k)^2T^1_2-2(1-T)W^kV^k(q^k)'\Psi_{RT}] \\
&+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}J^k_{X_iX_j}h_{ij}+\frac{1}{2}J^k_{TT}T^2_1+[(1-T)W^k(q^k)'\Psi_{RT}-V^k\Psi_{TX}]J^k_{XW}+\Psi_{TX}J^k_{XT} \\
&+J^k_{WT}[(1-T)W^k(q^k)'\Psi_{RT}-V^kT^2_1],
\end{aligned}
\tag{9}
$$

where q^k is an n-dimensional vector of the fraction of wealth of investor k invested in risky assets at time t, r is an n-dimensional vector of riskless rate, R is an n-dimensional vector of the instantaneous conditional expected percentage change in prices of risky assets per unit of time, is an m-dimensional vector of conditional means of state variables, h_{ij} is a covariance between state variables X_i and X_j, Ψ is an n-dimensional covariance matrix of the rates of return on risky assets, Ψ_{RX} is an n x m covariance matrix between asset returns and state variables, Ψ_{RT} is an n-dimensional vector of covariance between asset returns and tax rates, $\Psi_{TX'}$ is an m-dimensional covariance vector between taxes and state variables, and m-dimensional vectors, J^k_X, J^k_{XW}, and J^k_{XT} are the first derivative with respect to the state variables, cross partial derivatives with respect to the state variables and the wealth of investor k at time t, and cross partial derivatives with respect to the state variables and tax rates of the function $J^k(W^k,X,T,t)$, respectively.

The first order condition of equation (9) can be shown as

$$U^k_c = J^k_W, \tag{10}$$

$$
\begin{aligned}
&(1-T)J^k_W(R-r)-[J^k_W+(1-T)J^k_{WW}V^k-(1-T)J^k_{WT}]\Psi_{RT} \\
&\quad +(1-T)^2W^kJ^k_{WW}\Psi q+(1-T)\Psi_{RX}J^k_{XW}=0
\end{aligned}
\tag{11}
$$

From equation (11), the demand for the risky assets by investor k is derived as

$$W^kq^k=\frac{1}{(1-T)^2}\left[(1-T)\frac{J^k_{WW}}{-J^k_{WW}}\Psi^{-1}(R-r)+(1-T)\Psi_{-1}\Psi_{RX}\frac{J^k_{XW}}{-J^k_{WW}}\right. \tag{12}$$

$$-\Psi^{-1}\left(\frac{J^k_{WW}}{-J^k_{WW}}-(1-T)V^k+(1-T)\frac{J^k_{WT}}{J^k_{WW}}\right)\Psi_{RX}\Bigg].$$

Premultiplying both sides of equation (12) by produces

$$W^k\Psi q^k=\frac{1}{(1-T)^2}\Bigg[(1-T)\frac{J^k_{WW}}{-J^k_{WW}}(R-r)+(1-T)\Psi_{RX}\frac{J^k_{XW}}{-J^k_{WW}} \tag{13}$$
$$-\left(\frac{J^k_{WW}}{-J^k_{WW}}-(1-T)V^k+(1-T)\frac{J^k_{WT}}{J^k_{WW}}\right)\Psi_{RX}\Bigg].$$

Using equation (10), equation (13) can be rewritten as

$$W^k\Psi q^k\frac{\partial c^k}{\partial W^k}=\frac{1}{(1-T)^2}\Bigg[(1-T)\frac{U^k_c}{-U^k_{cc}}(R-r)+(1-T)\Psi_{RX}\frac{\partial c^k}{\partial X} \tag{14}$$
$$-\left(\frac{U^k_c}{-U^k_{cc}}-(1-T)V^k\frac{\partial c^k}{\partial W^k}+(1-T)\frac{\partial c^k}{\partial T}\right)\Psi_{RX}\Bigg].$$

Since the optimal consumption rate is a function of variables W, X, and T,

$$dc^k=\frac{\partial c^k}{\partial W^k}dW^k+\frac{\partial c^k}{\partial X}dX+\frac{\partial c^k}{\partial T}dT. \tag{15}$$

Now, $dW^k=(1-T)dV^k-V^kdT$+deterministic term.[4] Therefore,

$$\Psi^K_{Rc}=\Psi^K_{RW}\frac{\partial c^k}{\partial W^k}+\Psi_{RX}\frac{\partial c^k}{\partial X}+\Psi_{RT}\frac{\partial c^k}{\partial T}$$
$$=[(1-T)\Psi^K_{RV}-V^k\Psi_{RT}]\frac{\partial c^k}{\partial W^k}+\Psi_{RX}\frac{\partial c^k}{\partial X}+\Psi_{RT}\frac{\partial c^k}{\partial T}, \tag{16}$$

where Ψ^K_{Rc} is an n-dimensional covariance vector between the rates of return on risky

4) See Breeden (1979) for a detailed discussion.

assets and the investor's consumption rate and Ψ^{K}_{RV} is an n-dimensional covariance vector between the rates of return on assets and the taxable income of investor k.

Substituting equation (16) into equation (14) and rearranging, it follows that

$$-\frac{U^{k}_{c}}{-U^{k}_{cc}}(R-r)=\Psi^{K}_{Rc}-\frac{1}{(1-T)}\left(\frac{U^{k}_{c}}{U^{k}_{cc}}\right)\Psi_{RT}. \tag{17}$$

Aggregation over all investors results in

$$R_{j}-r=\frac{1}{N}\left[\mathrm{Cov}(R_{j}, c^{*})+\frac{1}{1-T}\mathrm{Cov}(R_{j}, T)\right], \tag{18}$$

where R_j is the expected rate of return on jth asset, c* is the aggregate changes in the consumption rate, and $N=\Sigma_{k}[-(U^{k}_{c}/U^{k}_{cc})]$, i.e. the aggregation of the inverse of the absolute risk aversion over investors.

As shown in equation (18), in equilibrium, the expected excess rate of return is determined by two factors. One is the covariance of the asset return with the aggregate changes in the consumption rate. To determine a martingale pricing process for the intertemporal asset pricing model when the investment opportunity set is stochastic, it is convenient to express the equilibrium in terms of consumption. Based on the idea that the covariance with consumption would be the appropriate sufficient statistic to measure the priced risk of a single security, Breeden (1979) developed a consumption-based asset pricing model wherein pricing of an asset depended on covariances with aggregate consumption rather than any market index or portfolio. The first term on the right hand side in equation (18) is identical to the consumption risk premium derived by Breeden in the absence of taxes.

The other factor which has an influence on asset returns is the covariance of asset returns with tax rate changes. As shown in equation (18), the expected excess rate of return is an increasing function of the covariance of asset returns with the tax rates and the impact of tax rates on asset returns will be greater when the current tax rate is relatively high. In their seminal paper, Mehra and Prescott (1985) demonstrated that the unusually large equity premium over short-term default-free debt rate over the ninety-year period 1889-1978 cannot be justified by general equilibrium models that abstract from transactions costs, liquidity constraints, and other market frictions. To account for the puzzling equity premium, many other explorations have been made by incorporating the possibilities of market crashes (Rietz (1988)), by introducing habit formation (Constantinides (1990)), and by considering the interaction between nonexpected utility and asymmetric market fundamentals (Hung (1994)). In this paper, the tax premium shown in equation (18) can partially explain the observed large equity risk premium.

The economic intuition of tax premium is as follows. The expected after tax payoff is

$$\begin{aligned} E[(P_{j1}-P_0)(1-T)] &= E(P_{j1}-P_0-TP_{j1}+TP_0) \\ &= E(P_{j1})-P_0-E(TP_{j1})+E(T)P_0 \\ &= E(P_{j1})-P_0+E(T)P_0-E(T)E(P_{j1})-Cov(P_{j1}, T). \end{aligned} \tag{19}$$

Since $Cov(R_j,T)=Cov[(P_{j1}-P_0)/P_0,T]=Cov(P_{j1},T)/P_0$, equation (19) shows that, as long as $Cov(R_j,T) > 0$, then the expected after tax payoff is lower than the case of no covariance with tax rates. Thus, there should be a compensation in terms of excess returns. Note that if the tax rate is never changing in the economy, equation (18) becomes

$$R_j-r=\frac{1}{N}Cov(R_j, c^*), \tag{20}$$

which is identical to the consumption-based capital asset pricing model derived by Breeden (1979). Thus, the single consumption beta model is a special case of the model presented in this paper.

Ⅳ. Empirical Implications

As discussed in the previous section, there will be a positive tax premium for a risky asset as long as the return on assets and the tax rate is positively correlated. Presumably, the effective tax rates tend to be high when the real output in the economy is high, and thus the rates of return on assets are high. Therefore, the omission of tax premium will understate the expected excess rate of return. However, whether there is a positive or a negative relationship between asset returns and effective tax rates is still an empirical issue.

There are papers which attempt to explain the predictability of asset returns. Bollerslev, Engle, and Wooldridge (1988), Harvey (1989), Ng (1991), Bodurtha and Mark (1991), and Chan, Karolyi, and Stulz (1992) investigate the role of changing betas within the context of the conditional CAPM. However, in these papers, the market is the sole risk factor. To overcome the problem, Engle and Rothschild (1990), Shanken (1990) and Evans (1994) recently examine multiple risk factors that can account for possible variations in the investment opportunity set that are ignored by the conditional CAPM. Since the general equilibrium effects of taxes on the asset prices seem to be well understood, the model developed in this paper can be used in examining the impact of taxes on the predictability of asset returns.

Equation (18) can also be used to examine the difference in the expected excess rates of return between different countries. The discrepancy in tax structures across different countries may partially explain the difference in the expected excess rates of return among countries. Thus far, there are few papers which formally incorporate the effects of tax structures of different countries into the international asset pricing analysis. For example, Roll (1992) compared Stock Price Indices across countries and proposed three factors which can explain disparate behavior of them; a technical aspect of index construction, industrial structure, and exchange rates. Even though, he did not consider the difference in tax structures across countries, tax structures may explain a significant portion of the difference in asset returns across countries.

Ⅴ. Conclusions

This paper analyzes the effects of stochastic tax rates on asset returns in the context of continuous time intertemporal asset pricing model. The result shows that as long as the asset returns and the tax rates are correlated, the model developed in the absence of taxation systematically misspecifies the excess rates of return on risky assets. Theoretically, the model developed in this paper can partially explain the puzzling large equity risk premium. Empirically, the tax premium can be one of the important factors to be considered in multiple risk factor capital asset pricing models.

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