Journal of the Korean Institute
of Industrial Engineers
Vol.20, No.1, March., 1994.

Sufficient Condition for Existence of Solution Horizon in Undiscounted Nonhomogeneous Infinit Horizon Optimization Problems

Yun Sun Park*, Myeon Sik Cho**

Abstract

Since many infinite horizon problems have infinite sequence of data to be considered, in general, it is impossible to express the optimal strategies finitely or to calculate them in finite time. This paper considers undiscounted nonhomogeneous determinitic infinite horizon problems. For those problems, we take a basic step to solve this class of infinite horizon problems optimally by giving a sufficient condition for a finite solution.

1. Introduction

Infinite horizon optimization is concerned with selecting an infinite sequence of decisions to optimize an infinite horizon problem over unbounded time.

Since many infinite horizon problems have infinite sequences of data to be considered, the optimal strategies cannot be expressed finitely nor calculated in finite time. The obvious solution method, though not exact, is to truncate a sufficiently large but tractable finite time horizon and solve that finite horizon problem using techniques like dynamic programming or linear programming (see Denardo [4], Ross [9]). However, the corresponding finite horizon optimal value may have some error due to the approximation of the finite planning horizon. An improved solution procedure would be to assign a salvage value, which represents the value obtainable from that time on, at the end of the finite horizon.

* Department of Industrial Engineering, Myongji University
** Department of Industrial Engineering, Kyunggi University

However, again, the salvage value itself cannot be estimated exactly. Consequently, in general, it is not possible to have an optimal sequence of decisions by the finite approximation of an infinite horizon problem.

However, there are some cases, as in the homogeneous Markov Decision Processes, in which an infinite horizon problem can be solved optimally using a finite method due to the special structure of the problem (see Hajnal [6], Ross [10][11]). Even when an infinite horizon problem is not homogeneous, occasionally, we can solve the infinite horizon problem optimally by a finite procedure (Bean and Smith [3], Alden and Smith [1]). If a current decision does not affect future desicions, the current decision can be obtained without relying on data in the far future. Thus, by selecting an appropriate finite horizon which separates the current and future decisions, we can obtain and initial optimal decision. This finite horizon which decouples the current and future decisions is called a *solution horizon*. After obtaining the initial decision using the solution horizon, the second, third, and other optimal decisions can be obtained by moving the horizon one step forward each time. Consequently, an infinite sequence of optimal decisions can be retrieved by repeating this finite procedure.

However, a solution horizon does not always exist in every nonhomogeneous infinite horizon problem. The necessary and most important condition for the existence of a solution horizon is the optimality of an *algorithmically* optimal strategy (see Hopp [7]). An algorithmically optimal strategy is an infinite horizon strategy defined to be an accumulation point strategy of finite horizon optimal strategies in the proper metric.

The reason that the optimality of an algorithmically optimal strategy is necessary for the existence of a solution horizon is the following. Since an algorithmically optimal strategy is an accumulation point of finite horizon optimal strategies, if the algorithmically optimal strategy is optimal and if it is unique, there exists a finite horizon beyond which the first optimal decision of finite horizon problems agrees with that of an algorithmically optimal strategy. However, that finite horizon is a solution horizon by definition and the algorithmically optimal strategy is optimal by the assumption. Thus, by solving that finite horizon (a solution horizon) problem, we can obtain the first optimal decision of an infinite horizon optimization problem. The purpose of this paper is to find the condition under which the algorithmically optimal strategy is optimal for undiscounted deterministic nonhomgeneous infinite horizon problems.

As discussed, to solve an infinite horizon optimization problem finitely, we need to show the existence of a solution horizon, which requires the optimality of an algorithmically optimal strategy. For the undiscounted nonhomogeneous Markov Decision Process, Hopp, Bean, and Smith [8] proved average optimality of an algorithmically optimal strategy when the Markov chains associated with the problem are weakly ergodic. For the discounted deterministic problem, which is a subset of infinite dimensional mathematical programming, Schochetman and Smith [12] showed that if a sequence of states can eventually be reached from any feasible state (the reachability condition), the accumulation point of the finite horizon optimal strategies up to that sequence of nodes is discounted optimal. Also, Bean and Smith [3] proved the same result under a weaker condition that a sequence of states can be reached weakly in discounted total cost (or reward) from an infinite horizon optimal strategy (they called it the weak reachability condition).

For the undiscounted deterministic problem, we will present a modified reachability condition to prove average optimality of an algorithmically optimal strategy. (refer to section 3 for the definition of reachability) Then, we will prove that if the sequence of nodes on an algorithmically optimal strategy is reachable from either an infinite horizon average optimal strategy or a finite horizon optimal strategy with uniformly bounded steps, an algorithmically optimal strategy is average optimal. Moreover, we will also prove that if the sequence of nodes on the path of an algorithmically optimal strategy is reachable from finite horizon optimal solution nodes with uniformly bounded steps, the finite horizon average optimal value approaches an infinite horizon optimal value (average optimal value convergence) and the average optimal value function is continuous at an algorithmically optimal strategy point (continuity of the average optimal value function).

Section 2 describes the undiscounted deterministic problem and introduces notation and definitions for this problem. In Section 3 we present the reachability condition to show various results including the average optimality of an algorithmically optimal strategy. Finally, Section 4 summarizes this paper with the application to the production and planning problem.

2. Problem Description and Notation

We consider the general infinite horizon sequential decision problem as in Bean and Smith [2] with a countably infinite directed graph, (N, A) with a single root node together with a real valued reward function $R:A \rightarrow R$. We refer to (N, A, R) as the decision network, arcs $(i, j) \in A$ as decisions, nodes $i \in N$ as states, and arc rewards $R(i, j)$ as decision rewards. We will also assume discrete time decision epochs.

Since there is a unique root node with in-degree zero, since we assume that each node has nonzero out-degree, each path can be feasibly continued to form a path covering the infinite horizon. A path is an infinite sequence of states (nodes) $(s_0, s_1, \cdots)$ where s_0 is the root node and $(s_n, s_{n+1}) \in A$ for all $n = 0, 1,$

2, ⋯, and S_n represents the set of feasible states at time n (the beginning of stage n). We refer to $(s_n, s_{n+1}, \cdots, s_N)$ as a (finite) path from s_n to s_N when $n < N$. The length of arc (s_n, s_{n+1}) associated with the decision (s_n, s_{n+1}) will also be referred to as a reward $R(s_n, s_{n+1})$. Sometimes, we will also denote t_n as a state at time (decision epoch) n.

We will define a strategy $x=(x_0, x_1, \cdots)$ as the infinite sequence of decisions $\{x_n\}_{n=0}^{\infty}$ along a path $\{s_n\}_{n=0}^{\infty}$ where $x_n=(s_n, s_{n+1})$ where $x_n=(s_n, s_{n+1})$, and we denote the set of feasible strategies by $X \subseteq \Pi_{n=0}^{\infty}$, where X_n is the set of feasible policies (decisions) at time n, and $x(n, s_n)$ is an element of $X(n, s_n)$. Then, $X(x) \in X$ is the set of all n-horizon feasible strategies, which can be defined as $\bigcup_{s_n \in S_n} X(n, s_n)$. Alos, we can define $X(n, B_n)$ as the set of feasible strategies leading to a set of states B_n, where $B_n \subseteq S_n$. Throughout this paper, superscript, *, on any strategy will represent that the strategy is optimal among the smallest class of strategies to which it belongs. For example, X^* is the set of average optimal infinite horizon strategies.

If a strategy x is used and the one period discount factor is $0<\alpha\leq 1$, the net present value at time k(the beginning of stage k) of the rewards from time k throught time N, $N>k$, is written $V_k(x;N)$. Note that in evaluating $V_k(x;N)$, the first k policies (decisions) of x are ignored. In general, we are interested in the value function from time 0 onward, which is written :

$$V_0(x;N)=\sum_{n=0}^{N-1} \alpha^n R(x_n),$$

where $R(x_n)=R(s_n, s_{n+1})$ and x passes along a path $\{s_n\}_{n=0}^{\infty}$. In an infinite horizon problem with discount factor α, $0<\alpha<1$, define x^* to be an (infinite horizon) α-discounted optimal strategy if

$$\lim_{N\to\infty} V_0(x^*;N)-\lim_{N\to\infty} V_0(x;N)\geq 0, \text{ for all } x\in X.$$

This definition is valid if the limit exists. It is possible that $V_0(x;N)$ diverges with N especially when $\alpha=1$. In that case, we define x^* to be an infinite horizon average optimal strategy if

$$\liminf_{N\to\infty} \frac{V_0(x^*;N)}{N}-\liminf_{N\to\infty}\frac{V_0(x;N)}{N}\geq 0, \text{ for all } x\in X.$$

We assume that $R(x_n)\leq \overline{R} < \infty$ for all n and x so that this liminf always exists.

We define the metric, ρ, between two feasible strategies x and x' as

$$\rho(x, x') = \sum_{k=0}^{\infty} 2^{-k}\Phi_k(x, x'),$$

where

$$\Phi_k(x, x') = \begin{cases} 0 & \text{if } x_k=x'_k \\ 1 & \textit{otherwise} \end{cases}$$

Let the set of feasible strategies, X, be compact in the topology introduced by the metric ρ. Under this topology, a sequence of strategies, $\{x^N\}_{N=0}^{\infty}$ with $x^N \in X$ for all N, converges to $x \in X$ if and only if its components, $\{x_k^N\}_{N=0}^{\infty}$, converge to x_k for all $k=0, 1, \cdots$. That is, $x^N \to x$ in ρ-metric as $N \to \infty$ if and only if, for all k, there is an N_k such that $x_k^N = x_k$ for $N > N_k$. This metric says that two strategies are closer if they agree over more initial decisions. By this metric, we wish to put more weight on earlier policy agreements since we need to have only the first few optimal decisions right now.

Based on the ρ-metric, we define an algorithmically optimal strategy, $\hat{x}$, and an accumulation point of finite horizon optimal strategies to a given sequence of nodes $\{s_N\}_{N=0}^{\infty}$, $\bar{x}$. Then, $\{\hat{s}_N\}_{N=0}^{\infty}$ and $\{\bar{s}_N\}_{N=0}^{\infty}$ are the corresponding sequences of nodes on paths of those strategies respectively. Mathematically :

Definition A strategy, $\hat{x}$, is an algorithmically optimal strategy if for some subsequences of integers $\{N_m\}_{m=0}^{\infty}$,

$$x^*(N_m) \to x \text{ as } m \to \infty \text{ in } \rho\text{-metric.}$$

$\{\hat{s}_N\}_{N=0}^{\infty}$ is the sequence of nodes on the path of $\hat{x}$:

Definition A strategy, $\bar{x}$, is an accumulation point strategy of finite horizon optimal strategies to a given sequence of nodes $\{s_N\}_{N=0}^{\infty}$ if for some subsequences of integers $\{N_m\}_{m=0}^{\infty}$

$$x^*(N_m, s_{N_m}) \to \bar{x} \text{ as } m \to \infty \text{ in } \rho\text{-metric.}$$

$\{\bar{s}_N\}_{N=0}^{\infty}$ is the sequence of nodes on the path of $\bar{x}$

Note that an algorithmically optimal strategy, $\hat{x}$, is a special case of an accumulation point strategy, $\bar{x}$.

3. Reachability

The main goal of this paper is to find a condition that is sufficient for the average optimality of an algorithmically optimal strategy. For discounted infinite dimensional mathematical programming, Schochetman and Smith [12] proved that if a sequence of nodes can be reached from all feasible states, the accumulation point, $\bar{x}$, is discounted optimal. Motivated by this, in this section, we will adopt the following reachability condition to show the average optimality of an algorithmically optimal strategy for undiscounted deterministic problems. First, we introduce the definition of reachability for our undiscounted deterministic problems.

Definition *(Reachability)* A given sequence of states $\{s_N\}_{N=0}^{\infty}$, $s_N \to \infty$, is reachable from a sequence of states $\{t_{j(N)}\}_{N=0}^{\infty}$ if there exists N' sufficiently large such that for each $N \geq N'$ there exists a sequence of decisions $(x_1^N, \cdots, x_N^N)$

1. which is feasible for N-horizon problem and achieves state s_N, i.e.,

$$(x_1^N, \cdots, x_N^N) \subseteq X(N, s_N),$$

2. whose first $j(N)$ decisions are feasible for a $j(N)$-horizon problem and achieves state $t_{j(N)}$, i.e.,

$$(x_1^N, \cdots, x_{j(N)}^N) \subseteq X(j(N), t_{j(N)}),$$

where $j(N)$ is a function of N such that $j(N) < N$.

We will employ the above definition of reachability to prove the average optimality of an algorithmically optimal strategy. However, the reachability condition of Schochetman and Smith [12] for the discounted problem alone was not sufficient for the discounted optimality of an algorithmically optimal strategy. The existence of an interest rate was also required for the optimality of an algorithmically optimal strategy. Thus, in undiscounted deterministic problems, by the absence of a discount factor, the reachability condition alone seems to be insufficient to prove the average optimality of an algorithmically optimal strategy. Moreover, our definition of reachability is weaker than the definition of reachability in Schochetman and Smith [12]. Ours requires a single node to be reachable from a node whereas theirs requires a whole sequence of nodes to be reachable from a node. Thus, we need an additional assumption of uniform boundedness to support reachability in the undiscounted case.

Assumption 1 When N is sufficiently large, there exists a subsequence $\{j(N)\}_{N=0}^{\infty}$ such that the sequence of nodes $\{\hat{s}_N\}_{N=0}^{\infty}$ is reachable from the sequence of finite horizon optimal solution nodes, $\{s_{j(N)}^{*}\}_{N=0}^{\infty}$, and $N-j(N)$ is uniformly bounded by a finite number, B, over all N, where $s_{j(N)}^{*}$ is a node which is achieved by $x^*(j(N))$ at time $j(N)$.

Now, we are ready to state our main theorem following the lemma below.

Lemma 1 (Continuity) Under Assumption 1, the average optimal value function is continuous at $\hat{x}$. That is, when

$$x^*(N_m) \rightarrow \hat{x} \text{ as } m \rightarrow \infty \text{ in the } \rho\text{-metric},$$

$$\liminf_{N\to\infty} \frac{V_0(x^*(N);N)}{N} = \liminf_{N\to\infty} \frac{V_0(\hat{x};N)}{N}.$$

(Proof)

- By the reachability condition, there exists a time point N and the corresponding node $\hat{s}_N$ which can be reached from $s_{j(N)}^{*}$ by a strategy x^N.
- By Assumption 1, $N-j(N) \leq B < \infty$ over all N.
- since $x^*(N_m) \rightarrow \hat{x}$ in the ρ-metric as $m \rightarrow \infty$, there exists a time point N_m which makes

$$x_k^*(N_m) = \hat{x}_k \text{ for } k = 1, \cdots, N.$$

By the principle of optimality, $\hat{x}$ is an optimal strategy up to a node $\hat{s}_N$. Then,

$V_0(\hat{x};N) \geq V_0(x^*(j(N));J(N)) + V_{j(N)}(x^N;N)$ for all N.

Dividing the both sides by N.

$$\frac{V_0(\hat{x};N)}{N} \geq \frac{V_0(x^*(j(N));j(N)) + V_{j(N)}(x^N;N)}{N}$$

Taking lim inf on both sides,

$$\liminf \frac{V_0(\hat{x};N)}{N} \geq \liminf \frac{V_0(x^*(j(N));j(N)) + V_{j(N)}(x^N;N)}{N}$$

But, $N-j(N)$ are uniformly bounded by a finite number, B, over all N and rewards are uniformly bounded for all N. Thus,

$$\frac{V_{j(N)}(\hat{x};N)}{N} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Then,

$$\liminf \frac{V_0(\hat{x};N)}{N} \geq \liminf \frac{V_0(x^*(j(N));j(N))}{N}.$$

Since $N = j(N) + (N-j(N))$,

$$\liminf \frac{V_0(x;N)}{N} \geq \liminf \frac{V_0(x^*(j(N));j(N))}{j(N)+(N-j(N))}.$$

Dividing denominator and numerator of the righthand side by $j(N)$,

$$\liminf \frac{V_0(\hat{x};N)}{N} \geq \liminf \frac{\frac{V_0(x^*(j(N));j(N))}{j(N)}}{1+\frac{N-j(N)}{j(N)}}.$$

which is

$$\liminf \frac{V_0(\hat{x};N)}{N} \geq \liminf \frac{V_0(x^*(j(N));j(N))}{j(N)}.$$

But, $\liminf \frac{V_0(x^*(j(N));j(N))}{j(N)} \geq \liminf \frac{V_0(x^*(N));N}{N}$ by the definition of lim inf. Thus,

$$\liminf \frac{V_0(\hat{x};N)}{N} \geq \liminf \frac{V_0(x^*(N);N)}{N} \qquad (1)$$

The other inequality is obvious since $x^*(N)$ is an optimal strategy for N-horizon problem, i.e.,

$$\liminf \frac{V_0(x^*(N);N}{N} \geq \frac{V_0(\hat{x};N)}{N} \text{ for all } N.$$

Taking liminf on both sides,

$$\liminf \frac{V_0(x^*(N);N)}{N} \geq \liminf \frac{V_0(\hat{x};N)}{N} \qquad (2)$$

From inequalities (1) and (2),

$$\liminf \frac{V_0(\hat{x};N)}{N} = \liminf \frac{V_0(x^*(N);N}{N}.$$

The above lemma shows the continuity of the average optimal value function at an algorithmically optimal strategy, $\hat{x}$. In the discounted case, the sufficient condition for this continuity is the existence of an interest rate. In the undiscounted Markov Decision Process case, as in Hajnal [6], weak ergodicity gives this continuity. Based on this lemma, we can prove the main goal of our research.

Theorem 1

Under Assumption 1, an algorithmically optimal strategy, $\hat{x}$, is average optimal, i.e.,

$$\liminf_{N\rightarrow\infty} \frac{V_0(\hat{x};N)}{N} \geq \liminf_{N\rightarrow\infty} \frac{V_0(x;N)}{N} \text{ for all } x \text{ in}$$

X.

In terms of the lim sup set, under Assumption 1,

$$\limsup_{N\to\infty} \mathrm{X}^*(\mathrm{N}) \subseteq \mathrm{X}^*.$$

(Proof)

By the definition of x^* and the fact that $\hat{x} \in X$,

$$\liminf \frac{V_0(x^*;N)}{N} \geq \liminf \frac{V_0(\hat{x};N)}{N} \qquad (3)$$

Since $x^*(N)$ is an optimal solution for N-horizon problem,

$$\frac{V_0(x^*(N);N)}{N} \geq \frac{V_0(x^*;N)}{N}, \text{ for all } N$$

Taking lim inf on the both sides,

$$\liminf \frac{V_0(x^*(N);N}{N} \geq \liminf \frac{V_0(x^*;N)}{N}.$$

But the previous lemma says that

$$\liminf \frac{V_0(x^*(N);N)}{N} \geq \liminf \frac{\mathrm{V_0(x^*;N)}}{\mathrm{N}}.$$

Thus,

$$\liminf \frac{V_0(\hat{x};N)}{N} \geq \liminf \frac{V_0(x^*;N)}{N} \qquad (4)$$

Then, from inequalities (3) and (4),

$$\liminf \frac{V_0(x^*;N)}{N} = \liminf \frac{V_0(\hat{x};N)}{N}.$$

But, $\liminf \frac{V_0(x^*;N)}{N} \geq \liminf \frac{V_0(x;N)}{N}$ for all x in X. Thus,

$$\liminf_{N\to\infty} \frac{V_0(\hat{x};N)}{N} \geq \liminf_{N\to\infty} \frac{V_0(x;N)}{N} \text{ for all } x \text{ in } X.$$

The fact that an algorithmically optimal strategy is average optimal leads us to compute infinite horizon problems finitely. Moreover, the next corollary justifies the finite horizon approximation of infinite horizon problems.

Corollary 1 (average optimal value convergence) Under Assumption 1, the finite horizon average optimal value converges to the infinite horizon average otpimal value, that is,

$$\liminf \frac{V_0(x^*(N);N)}{N} = \liminf \frac{V_0(x^*;N)}{N}.$$

(Proof)

From the Lemma 1 and and the Theorem 1.

The above corollary says that solving sufficiently long finite horizon problem is a reasonable approximation for the infinite horizon problem.

Now, we will introduce a different assumption using an accumulation point strategy, $\bar{x}$, and the corresponding sequence of nodes $\{\bar{s}_N\}_{N=0}^{\infty}$ to obtain a more general result than Theorem 1. However, we will lose the continuity result in Lemma 1 and the convergence result in Corollary 1.

Assumption 2 When N is sufficiently large, there exists a subsequence $\{j(N)\}_{n=0}^{\infty}$ such that the sequence of nodes $\{\bar{s}_N\}_{N=0}^{\infty}$ is reachable from the sequence of infinite horizon optimal solution nodes, $\{s^*_{j(N)}\}_{N=0}^{\infty}$ and $N-j(N)$ is uniformly bounded by a finite number, B, over all N, where $s^*_{j(N)}$ is a node on the path of an infinite horizon average optimal strategy, x^*.

Note that Assumption 1 required that the sequence of nodes $\{\hat{s}_N\}_{N=0}^{\infty}$ is reachable from the sequence of finite horizon optimal solution nodes, $\{s^{*\prime}_{j(N)}\}_{N=0}^{\infty}$.

Theorem 2

Under Assumption 2, the accumulation point, $\bar{x}$, is an average optimal strategy, i.e.,

$$\liminf_{N\to\infty} \frac{V_0(\bar{x};N)}{N} \geq \liminf_{N\to\infty} \frac{V_0(x;N)}{N} \text{ for all } x \text{ in } X.$$

(Proof)

By Assumption 2, there eists an infinite horizon average optimal node $s^*_{j(N)}$ from which $\bar{x}_N$ can be reached through a strategy x^N, and $N-j(N)$ is uniformly bounded over all N. Also, by the principle of optimality, $\bar{x}$ is the optimal strategy up to all nodes, $\{\bar{s}_N\}_{n=0}^{\infty}$. Then,

$$V_0(\bar{x};N) \geq V_0(x^*(j(N));s^*_{j(N)}),\ j(N)) + V_{j(N)}(x^N;N).$$

But, $V_0(x^*(j(N));s^*_{j(N)}),\ j(N)) = V_0(x^*;j(N))$ by the principle of optimality and the fact that $s^*_{j(N)}$ is on the optimal path. Thus,

$$V_0(\bar{x};N) \geq V_0(x^*;j(N)) + V_{j(N)}(x^N;N).$$

Dividing the both sides by N gives

$$\frac{V_0(\bar{x};N)}{N} \geq \frac{V_0(x^*;j(N)) + V_{j(N)}(x^N;N)}{N}.$$

Taking lim inf on both sides,

$$\liminf \frac{V_0(\bar{x};N)}{N} \geq \liminf \frac{V_0(x^*;j(N)) + V_{j(N)}(x^N;N)}{N}$$

But $N-j(N)$ are uniformly bounded by finite number over N and rewards are uniformly bounded for all N. Thus,

$$\frac{V_{j(N)}(\bar{x};N)}{N} \to 0 \text{ as } N \to \infty.$$

Then,

$$\liminf \frac{V_0(\bar{x};N)}{N} \geq \liminf \frac{V_0(x^*;j(N))}{N}$$

Since $N = j(N) + (N - j(N))$,

$$\liminf \frac{V_0(\bar{x};N)}{N} \geq \liminf \frac{V_0(x^*;j(N))}{j(N) + (N-j(N))}$$

Dividing denominator and numerator of the righthand side term by $j(N)$,

$$\liminf \frac{V_0(\bar{x};N)}{N} \geq \liminf \frac{\frac{V_0(x^*;j(N))}{j(N)}}{1 + \frac{N-j(N)}{j(N)}}$$

which is

$$\liminf \frac{V_0(\bar{x};N)}{N} \geq \liminf \frac{V_0(x^*;j(N))}{j(N)}$$

But, $\liminf \frac{V_0(x^*;j(N))}{j(N)} \geq \liminf \frac{V_0(x^*;N)}{N}$ by the definition of lim inf. Thus,

$$\liminf \frac{V_0(\bar{x};N)}{N} \geq \liminf \frac{V_0(x^*;N)}{N}$$

Since x^* is average optimal,

$$\liminf \frac{V_0(x^*;N)}{N} \geq \liminf \frac{V_0(x;N)}{N} \quad \text{for all } x \in X.$$

Thus,

$$\liminf \frac{V_0(\bar{x};N)}{N} \geq \liminf \frac{V_0(x;N)}{N} \quad \text{for all } x \in X.$$

In summary, under Assumption 1, we obtained three important results: continuity of average optimal value function at an algorithmically optimal strategy, x, the average optimality of an x, and average optimal value convergence. Even though Assumption 2 gives only one result, the fact that the accumulation point strategy, $\bar{x}$, is average optimal is more general since an algorithmically optimal strategy, $\hat{x}$, is included in the set of accumulation point strategies of finite horizon optimal strategies to any sequence of nodes.

4. Conclusion

This paper took a basic step for the finite solution of undiscounted deterministic infinite horizon optimization problems by proving the average optimality of an algorithmically optimal strategy. The reachability condition with the uniform boundedness assumption turned out to be a sufficient condition for the existence of a solution horizon which make it possible to solve infinite horizon problems finitely using the notion of the solution horizon and algorithmically optimal strategy. Under these conditions we also proved average optimal value convergence and continuity of the average optimal value function at an algorithmically optimal strategy. The above result can be applied to the production planning problem in Schochetman and Smith [12].

References

[1] Alden, J., R. Smith [1987], "Rolling Horizon Procedures in Nonhormogeneous Markov Decision Processes," Technical Report 87-25, Department of Industrial and Operations Engineering, The University of Michigan, Ann Arbor, MI 48109.

[2] Bean, J., R. Smith [1984], "Conditions for the Existence of Planning Horizons," *Mathematics of Operations Research*, Vol. 9, 1984, pp. 391-401.

[3] Bean, J., R. Smith [1986], "Conditions for the Discovery of Solution Horizons," Techical Report 86-23, Department of Industial and Operations Engineering, The University of Michigan, Ann Arbor, Michigan 48109.

[4] Denardo, E. [1982], *Dynamic Programming: Models and Application*, Prentice-Hall, Inc., Englewood Ciffs, New Jersey.

[5] Derman, C. [1966], "Denumerable State Markov Decision Process," *Annals of*

Mathematical statistics, Vol. 37, pp. 1545-1554.

[6] Hajnal, J. [1958], "Weak Ergodicity in Nonhomogeneous Markov Chains," *Preceedings of the Cambridge Philosophical Society*," Vol. 52, pp. 67-77.

[7] Hopp, W. [1988], "Identifying Forecast horizons in Nonhomogeneous Markov Decision Processes," *Operations Research*, Vol.37, pp. 339-343.

[8] Hopp, W., J. Bean and R. Smith [1987], "A New Optimality Criterion for Nonhomogeneous Markov Decsion Processes," *Operations Research*, Vol. 35, pp. 875-883.

[9] Ross, S. '[1983], *Introduction to Stoshastic Dynamic Programming*, Academic Press Inc., New York, New York.

[10] Ross, S. [1971], "On the Nonexistence of ε-optimal Randomized Stationary Policies in Average Cost Markovian Decision Models," *Annals of Mathematical Statistics*, Vol. 42, pp. 1767-1768.

[11] Ross, S. [1968], "Non-Discounted Denumerable Markov Decision Models," *The Annals of Mathematical Statistics*, Vol. 39, p. 412-423.

[12] Schochetman, I., R. Smith [1989], "Infinite Horizon Optimization," *Mathematics of Operations Research*, Vol. 14, pp. 559-574.