▩ 연구논문

Bayes Estimation of Reliability in the Strength-Stress Models+

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Abstract

We obtain the Bayes estimator(BE), the minimum variance unbiased estimator (MVUE) and maximun likelihood estimator(MLE) of the reliability when the distribution of the stress and the strength are Weibull with known shape parameters. The experiment is terminated before all of the items on the test have failed and the failed items are partially replaced. Performance of the three estimators for moderate size samples are compared through Monte Carlo simulation.

1. Introduction

Let X be the strength of the unit and Y be the stress placed on the unit by the operating environment in the strength-stress model. An unit is able to perform its intended function if its strength is greater than the stress imposed upon it. We define the reliability, θ, as the probability that the unit performs its task satisfactorily. If the strength X is distributed continuously with $F(x)$, the stress Y is also distributed continuously with $G(y)$, then the reliability is

$$\begin{aligned} \theta = P(Y<X) &= \int P(Y<X \mid X=x)dF(x) \\ &= \int G(x)dF(x). \end{aligned} \quad (1.1)$$

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The problem of estimating θ, first considered by Birnbaum (1956) has found an increasing number of applications. For other related studies on this problem, we see Mazumder (1970), Church and Harris (1970), Enis and Geisser (1971), Johnson (1988) and Weerahandi and Johnson(1992), among others.

In the case of a complete sample is not available the problems of estimating θ have been also studied for the censored data in strength-stress model. The practice of terminating a life test with only partial information available is called censoring.

In this paper, the Bayesian estimator (BE), maximum likelihood estimator (MLE) and minimum variance unbiased estimator (MVUE) of θ are considered in the failure censored case with partial replacement. When n items are placed on life test and the first k that fail are immediately replaced but subsequent failures are not replaced, we call it the partial replacement procedure. The experiment is terminated when the r^{th} item fails so that, for a partial replacement, $k+1 < r < n+k$.

In section 2, we are concerned with the BE, MLE and MVUE of θ to the failure consored case with partial replacement. In section 3, performance of the three estimators for moderate sized samples are compared through Monte Carlo simulation.

2. Estimation of θ

2.1 Bayes estimator of θ

Let $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_m$ be independently and identically distributed as

$$f_1(x \mid \alpha, p_1) = \alpha p_1 x^{p_1 - 1} \exp(-\alpha x^{p_1}), \quad x \geq 0, \alpha, p_1 > 0 \qquad (2.1)$$

and

$$f_2(y \mid \beta, p_2) = \beta p_2 y^{p_2 - 1} \exp(-\beta y^{p_2}), \quad y \geq 0, \beta, p_2 > 0 \qquad (2.2)$$

respectively. In Weibull reliability analysis it is not unusual that the value of the shape parameter is known. We assume that the shape parameters are known and equal to p, i.e, $p_1 = p_2 = p$. It is well known that, if T has a Weibull distribution $W(\lambda, p)$, then T^p follows an exponential distribution. Since X and Y are independent

$$\theta = Pr(Y<X) = \int_0^{\infty}\int_0^{x} I(Y, X)dF_2(y)dF_1(x)$$

$$= \frac{\beta}{\alpha+\beta}, \tag{2.3}$$

where $I(Y, X) = \begin{cases} 1 & \text{if } Y<X \\ 0 & otherwise. \end{cases}$

Consider the procedure that n and m items are placed on life test and the first k_1 and k_2 that fail are immediately replaced but subsequent failures are not replaced, respectively. The experiments are terminated when r and s items fail, i.e, $k_1+1 < r < n+k_1$, and $k_2+1 < s < m+k_2$, respectively. $X_{(1)}, X_{(2)}, \cdots, X_{(k_1)}, \cdots, X_{(r)}$ and $Y_{(1)}, Y_{(2)}, \cdots, Y_{(k_2)}, \cdots, Y_{(s)}$ be r and s ordered failure times, respectively.

We assume that α and β are independent, a priori, and employ conjugate prior distributions for α and β,

$$q_1(\alpha) \propto \alpha^{\gamma_1 - 1} e^{\delta_1 \alpha}, \quad \gamma_1, \delta_1 > 0 \tag{2.4}$$

and

$$q_2(\beta) \propto \beta^{\gamma_2 - 1} e^{\delta_2 \beta}, \quad \gamma_2, \delta_2 > 0 \tag{2.5}$$

Since the likelihood of the sample of r and s observations from (2.1) and (2.2) are

$$L(\alpha) \propto \alpha^{r} \exp(-\alpha t_1), \tag{2.6}$$

where $t_1 = nx^{p}_{(k_1)} + \sum_{i=1}^{r-k_1} x^{p}_{(k_1+i)} + (n-r+k_1)x^{p}_{(r)}$

and,

$$L(\beta) \propto \beta^{s} \exp(-\beta t_2), \tag{2.7}$$

where $t_2 = my^{p}_{(k_2)} + \sum_{i=1}^{s-k_2} y^{p}_{(k_2+i)} + (m-s+k_2)y^{p}_{(s)}$

respectively, using (2.4) and (2.5), we obtain the posterior densities for α and β as follows;

$$\pi_1(\alpha) \propto \alpha^{(r+\gamma_1-1)} e^{-\alpha(t_1+\delta_1)} \tag{2.8}$$

and

$$\pi_2(\beta) \propto \beta^{(s+\gamma_2-1)} e^{-\beta(t_2+\delta_2)}. \tag{2.9}$$

If we let $\theta = \frac{\beta}{\alpha+\beta}$ and $w=\alpha+\beta$, using (2.8) and (2.9), we obtain the joint posterior of θ and w as follows

$$g_{1,2}(\theta, w) \propto \omega^{r+s+\gamma_1+\gamma_2-1} \theta^{s+\gamma_2-1}(1-\theta)^{r+\gamma_1-1}$$
$$\exp[-w(t_1+\delta_1)(1-c\theta)], \tag{2.10}$$

where $c = 1-\frac{t_2+\delta_2}{t_1+\delta_1} < 1$.

Integrating out ω in (2.10), we obtain the marginal posterior density of θ,

$$g_1(\theta) \propto \theta^{s+\gamma_2-1}(1-\theta)^{r+\gamma_1-1}[(t_1+\delta_1)(1-c\theta)]^{-(r+s+\gamma_1+\gamma_2)}. \tag{2.11}$$

For the estimation of θ, we may use the posterior mean of θ given by

$$\theta^* = E(\theta) = \frac{s+\gamma_2}{r+s+\gamma_1+\gamma_2}(1-c)^{s+\gamma_2}$$
$$\cdot {}_2F_1(r+s+\gamma_1+\gamma_2, s+\gamma_2+1, r+s+\gamma_1+\gamma_2+1; c), \ |c| < 1 \tag{2.12}$$

where ${}_2F_1(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}dt.$

If we let $\rho = \frac{1-\theta}{1-c\theta}$, we obtain

$$f(\rho) \propto \rho^{r+\gamma_1-1}(1-\rho)^{s+\gamma_2-1}, \quad 0 < \rho < 1 \tag{2.13}$$

Thus for $0 \le \rho_1 < \rho_2 \le 1$,

$$Pr[\rho_1 < \rho < \rho_2] = I(\rho_2; r+\gamma_1, s+\gamma_2) - I(\rho_1; r+\gamma_1, s+\gamma_2). \tag{2.14}$$

where $I(\rho : s, r) = \dfrac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} \int_0^{\rho} t^{s-1}(1-t)^{r-1}dt, \quad 0 \le \rho \le 1, \quad s, r > 0.$

Using (2.14) we obtain the limits on θ

$$Pr\left[\frac{1-\rho_2}{1-c\rho_2} < \theta < \frac{1-\rho_1}{1-c\rho_1} \mid c \right]$$
$$= I(\rho_2 ; r+\gamma_1, s+\gamma_2) - I(\rho_1 ; r+\gamma_1, s+\gamma_2) . \tag{2.15}$$

(2.15) can be used to obtain the probability that θ is within the profixed interval (θ_1, θ_2) as follows,

$$Pr(\theta_1 < \theta < \theta_2 \mid c) = I\left(\frac{1-\theta_1}{1-c\theta_1} ; r+\gamma_1, s+\gamma_2 \right) - I\left(\frac{1-\theta_1}{1-c\theta_2} ; r+\gamma_1, s+\gamma_2 \right) \tag{2.16}$$

In order to obtain the bound on the Bayes estimator, we note the following. If $z^* = \alpha / \beta > 0$, then

$$E(z^*) = \int_0^{\infty} \int_0^{\infty} z^* \pi_1(\alpha) \pi_2(\beta) d\alpha d\beta$$
$$= \frac{r+\gamma_1}{s+\gamma_2-1}(1-c)^{-1} . \tag{2.17}$$

Similary let $z = \beta / \alpha$

$$E(z) = \frac{s+\gamma_2}{r+\gamma_1-1}(1-c)^{-1} . \tag{2.18}$$

If we define $\theta = h(z^*) = (1+z^*)^{-1}$ then $h(z^*)$ is a convex function of z^* and if we define $\theta = g(z) = (1+z^{-1})^{-1}$ then $g(z)$ is a concave function of z. Thus by Jensen's inequality we obtain

$$h[E(z^*)] = \left[1 + \frac{r+\gamma_1}{s+\gamma_2-1}(1-c)\right]^{-1}$$
$$\le E(\theta) = \theta^*$$
$$\le \left[1 + \frac{r+\gamma_1-1}{s+\gamma_2}(1-c)\right]^{-1} = g[E(z)] . \tag{2.19}$$

As an alternative to the use of the subjective priors of (2.4) and (2.5), we may be inclined to employ so-called "vague" prior distributions for both α and β,

$$g_1(\alpha) = \frac{1}{\alpha^a} \quad \text{and} \quad g_2(\beta) = \frac{1}{\beta^b}.$$

It is easily seen that, for these priors, the results corresponding to those obtained using conjugate priors, may be obtained by putting $\delta_1 = \delta_2 = 0$, $\gamma_1 = 1-a$ and $\gamma_2 = 1-b$ in (2.8) $-$(2.18), and substituting

$$c^* = \frac{t_1 - t_2}{t_1} \tag{2.20}$$

for c in these expressions. In particular bound on $E(\theta)$ for this case is

$$[1 + \frac{r-a+1}{s-b}(1-c^*)]^{-1} < E(\theta) < [1 + \frac{r-a}{s-b+1}(1-c^*)]^{-1}. \tag{2.21}$$

2.2 MLE of θ

From the point view of sampling theory, it is clear that the MLE of $\alpha^* = \alpha^{-1}$ and $\beta^* = \beta^{-1}$ are

$$\hat{\alpha} = \frac{t_1}{r} \quad \text{and} \quad \hat{\beta} = \frac{t_2}{s}$$

Hence, by the invariant property of MLE, the MLE of θ is

$$\hat{\theta} = \frac{\hat{\alpha}^*}{\hat{\alpha}^* + \hat{\beta}^*}. \tag{2.22}$$

Since $2\alpha t_1$ and $2\beta t_2$ are independent and follow central χ^2 distribution with $2r$ and $2s$ degree of freedom, respectively, $U = \frac{\beta t_2}{\alpha t_1 + \beta t_2}$ follows a beta distribution with parameters s and r Thus for $u_1 < u_2$, limits on U such that

$$Pr[u_1 < U < u_2] = I(u_2; s, r) - I(u_1; s, r) \tag{2.23}$$

may, after some algebraic manipulations, be converted to limits on θ,

$$Pr\left[\frac{ru_1\hat{\theta}}{s(1-u_1)(1-\hat{\theta})+ru_1\hat{\theta}} < \theta < \frac{ru_2\hat{\theta}}{s(1-u_2)(1-\hat{\theta})+ru_2\hat{\theta}} \right]$$
$$= I(u_2\,;s,r) - I(u_1\,;s,r)\,. \tag{2.24}$$

Now, letting $\rho_1 = 1-u_2$, $\rho_2 = 1-u_1$ and noting

$$\hat{\theta} = \frac{s}{r(1-c^*)+s}, \tag{2.25}$$

where $c^* = \dfrac{t_1 - t_2}{t_1}$, we obtain that

$$Pr\left[\frac{1-\rho_2}{1-c^*\rho_2} < \theta < \frac{1-\rho_1}{1-c^*\rho_1} \right] = I(\rho_2\,;r,s) - I(\rho_1\,;r,s) \tag{2.26}$$

Hence by comparing (2.15) with (2.26), we can see that the MLE of θ based on confidence procedures are equivalent to the Bayesian estimator of θ based on "informationless" priors.

2.3 MVUE of θ

Sinha and Kale(1980) showed that α^* and β^* are complete sufficient statistics for α and β, respectively. Using the Blackwell-Rao and Lehmann-Scheffe Theorem, the unique MVUE of θ is obtained by taking the conditional expectation of $I(Y, X)$ given (α^*, β^*).

$$\tilde{\theta} = E[I(Y, X) \mid \hat{\alpha}^*, \hat{\beta}^*]$$
$$= \iint_A I(y, x) h(x, y)\, dx\, dy \tag{2.27}$$

where A is the simple space in which both $f(x \mid \alpha^*)$ and $g(y \mid \beta^*)$ are nonzero and

$$I(Y, X) = \begin{cases} 0 & \text{if } Y > X \\ 1 & \text{if } Y < X\,. \end{cases}$$

For the case of $t_2 < t_1$,

$$\tilde{\theta} = E[I(X, Y) \mid \hat{\alpha}^*, \hat{\beta}^*]$$

$$= \frac{(r-1)(s-1)}{t_1 t_2} \int_0^{t_2} \left(1-\frac{y}{t_2}\right)^{s-2} \cdot \left[\int_{y}^{t_1} \left(1-\frac{x}{t_2}\right)^{r-2} dx \right] dy$$

$$= (s-1) \int_0^1 (1-v)^{s-2} (1-zv)^{r-1} dv$$

$$= {}_2F_1\left(1-r, 1, s; \frac{t_2}{t_1}\right), \tag{2.28}$$

where $\frac{y}{t_2} = v$ and $\frac{t_2}{t_1} = z$.

3. Empirical comparision for moderate sized sample

In this chapter, we compare the relative performance of the Bayes estimator, MLE and MVUE of θ, for a moderate sized sample through Monte Carlo simulation. For $r=s=3(1)\ 10, 12, 15, 20$ and $n=m=20$, estimates of the mean square error (MSE) and bias are obtained from 2,000 trials with $\theta=0.5$, 0.666, 0.75, 0.8. The results on the estimated MSE and bias appear in the table. Although MVUE is known to be unbiased, its estimated bias is recorded as a check on the computation. From the results of the simulation, we know the following fact;

① When $\theta=0.5$ the estimated MSE's are not small for $r=s=3(1)\ 6$.

② The estimated MSE's of Bayes estimator are smaller than the others.

③ The estimated MSE's are influenced by the number of censoring.

〈 Table 〉 The Estimated MSE and Bias

θ	(r, s)	MSE			Bias		
		BE	MLE	MVUE	BE	MLE	MVUE
0.5	(3, 3)	0.02834	0.03543	0.04359	0.00033	0.00019	0.05808
	(4, 4)	0.02349	0.02811	0.89361	0.00107	0.00135	0.03575
	(5, 5)	0.02021	0.02349	0.05421	0.00169	0.00182	0.02222
	(6, 6)	0.01661	0.01892	0.06674	0.00001	0.00010	0.00755
	(7, 7)	0.01467	0.01649	0.01808	0.00016	0.00016	0.00270
	(8, 8)	0.01338	0.01485	0.01693	0.00175	0.00185	0.00156
	(9, 9)	0.01224	0.01345	0.01481	0.00339	0.00354	0.00335
	(10, 10)	0.01054	0.01149	0.01289	0.00140	0.00141	0.00169
	(12, 12)	0.00909	0.00979	0.01064	0.00290	0.00298	0.00310
	(15, 15)	0.00766	0.00814	0.00869	0.00223	0.00232	0.00241
	(20, 20)	0.00750	0.00785	0.00928	0.00333	0.00342	0.00279
0.6667	(3, 3)	0.27149	0.03163	0.04169	0.03744	0.02297	0.01693
	(4, 4)	0.02132	0.02403	0.02866	0.03020	0.01826	0.00076
	(5, 5)	0.01852	0.02053	0.02276	0.02461	0.01424	0.00061
	(6, 6)	0.01471	0.01599	0.01792	0.02135	0.01219	0.00063
	(7, 7)	0.01317	0.01420	0.01567	0.01912	0.01094	0.00089
	(8, 8)	0.01107	0.01182	0.01288	0.01560	0.008134	0.00084
	(9, 9)	0.01237	0.01331	0.01470	0.01713	0.00918	0.00063
	(10, 10)	0.00958	0.01008	0.01077	0.01466	0.00859	0.00148
	(12, 12)	0.00859	0.00897	0.00946	0.01163	0.00642	0.00051
	(15, 15)	0.00659	0.00683	0.00714	0.00895	0.00461	0.00022
	(20, 20)	0.00603	0.00621	0.00643	0.00432	0.00098	0.00264
0.75	(3, 3)	0.02573	0.02796	0.04757	0.05254	0.03243	0.00594
	(4, 4)	0.01836	0.01922	0.02118	0.04024	0.02343	0.00095
	(5, 5)	0.01504	0.01553	0.01618	0.03424	0.01999	0.00128
	(6, 6)	0.01205	0.01235	0.01299	0.02605	0.01349	0.00176
	(7, 7)	0.01026	0.01036	0.01073	0.02562	0.01454	0.00139
	(8, 8)	0.00934	0.00942	0.00969	0.02447	0.01467	0.00320
	(9, 9)	0.01006	0.01023	0.01064	0.02346	0.01271	0.00004
	(10, 10)	0.00755	0.00761	0.00779	0.01754	0.00943	0.00022
	(12, 12)	0.00614	0.00618	0.00630	0.01394	0.00697	0.00075
	(15, 15)	0.00511	0.00513	0.00521	0.01117	0.00550	0.00067
	(20, 20)	0.00494	0.00497	0.00505	0.00745	0.00315	0.00141
0.8	(3, 3)	0.02126	0.02133	0.02032	0.05615	0.03299	0.00271
	(4, 4)	0.01609	0.01596	0.01664	0.04228	0.02368	0.00021
	(5, 5)	0.01329	0.01302	0.01277	0.03858	0.02292	0.00361
	(6, 6)	0.01038	0.01010	0.01005	0.03096	0.01742	0.00148
	(7, 7)	0.00802	0.00775	0.00767	0.02552	0.01346	0.00038
	(8, 8)	0.00721	0.00697	0.00690	0.02284	0.01232	0.00026
	(9, 9)	0.00832	0.00808	0.00801	0.02531	0.01373	0.00039
	(10, 10)	0.00593	0.00577	0.00572	0.01837	0.00974	0.00015
	(12, 12)	0.00501	0.00486	0.00479	0.01772	0.01034	0.00234
	(15, 15)	0.00386	0.00386	0.00383	0.01179	0.00588	0.00050
	(20, 20)	0.00363	0.00359	0.00358	0.00775	0.00326	0.00146

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