▩ 연구논문

Efficiency Loss Due to Censoring for Testing against a Change-Point in Failure Rate

Junkeun Yum · Daekyung Kim
Dept. of Statistics, Dongguk University

Abstract

A frequently recurring question posed by researchers concerns a test of a constant failure rate against the alternative of a failure rate involving a single chang-point. Park (1988) has presented a new test procedure for testing constant failure rate against a bathtub shaped (upside-down bathtub shaped) failure rate, assuming that the proportion of population that fails at or before the change point of failure rate is known. Jeong (1992) has extended Park's test to randomly censored data under the same assumption. In this paper, we have investigated efficiency loss to the presence of censoring.

1. Introduction

Failure rate has been investigated extensively, particularly in reliability studies and actuarial science. If the failure distribution F has a density f, the failure rate function $r(t)$ is defined for those values of t for which $F(t) < 1$ by

$$r(t) = f(t)/\bar{F}(t)$$

where $\bar{F}(t) = 1 - F(t)$ is the survival function. If $r(t)$ increases monotonically over time, the life distribution is said to have increasing failure rate (IFR). If $r(t)$ decreases monotonically over time, the life distribution is said to have decreasing failure rate (DFR). If $r(t)$ is constant, the life distribution has constant failure rate (CFR). Another typical failure rate is a bathtub-shaped or U-shaped function. A life distribution has bathtub-shaped failure rate (BTR) at τ if $r(s) \geq r(t)$ for $0 \leq s \leq t < \tau$ and $r(s) \leq r(t)$ for $\tau \leq s \leq t < \infty$. It is a class of life distributions arising naturally in reliability situations. The BTR testing is useful for the many situations [Guess, Hollander and Proschan, 1986: Park, 1988].

Recently, Park (1988) presents a test procedure for testing CFR against a BTR in uncensored data, assuming that the proportion of population that fails at or before the change point of failure rate is known. Also, Jeong (1992) has extended Park's test to randomly censored data under the same assumption. They established the asymptotic normality of the test statistics by the L-statistic theory and calculated the asymptotic null variances. We are interested in efficiency loss due to censoring as censoring amount varies. In this paper, we calculate the efficiency loss due to censoring using their asymptotic null variances.

2. Efficiency Loss Due to Censoring

In this section we study the efficacy loss due to censoring by comparing the efficiency of Park (1988) test based on T^*_n for uncensored model with the efficacy of Jeong (1992) test based on T_n^c for randomly censored model.

Assuming that $p \equiv F(\tau)$ $\tau \equiv$ change point, is known, Park (1988) has considered asymptotic null variance of the test statistic T^*_n for BTR test as follows.

$$\sqrt{n}\, T^*_n \rightarrow N(0, \mu^2\sigma_0^2) \quad \text{as} \quad n \rightarrow \infty,$$

where $\sigma_0^2 \equiv \sigma^2(J, F) = (1/3) - p + p^2$

Jeong (1992) also has derived asymptotic distribution of test statistic for censored case using the Kaplan-Meier estimator (1958) and calculated asymptotic null variance.

$$\sqrt{n}\, T_n^c \rightarrow N(0, \mu^2\sigma_0^2(F, H)) \quad \text{as} \quad n \rightarrow \infty,$$

where

$$\sigma_0^2(F, H) = \int_0^1 [(1-p)(1-2p)I_{(0,p)}(t) - B(t)]^2 (1-t)^{-1} [\overline{L}(F^{-1}(t))]^{-1} dt.$$

In $\sigma_0^2(F, H)$, H is censoring distribution and $\overline{L}(t) = 1 - L(t) = \overline{F}(t)\overline{H}(t)$. Also $B(t) \equiv \int_t^1 J(u)\,du$

Since the T_n^c statistic reviewed in this paper is an extension of the T_n^* statistic of Park (1988) to accomodate the censored data, it is interesting to compare the power of the T_n^* test based on n observations in the uncensored case with the power of the T_n^c based on n' observations for the randomly censored case. Let F_θ be a parametric family within the BTR class with F_{θ_0} being exponential. Then we assume the randomly censored data with $F = F_\theta$ and with censoring distribution H. Consider a sequense of alternatives $\theta_n = \theta_0 + cn^{-1/2}$(with $c > 0$) tending to the null hypothesis, let $\beta_n(\theta_n)$ be the power of the approximate α-level T^*_n test based on n observations in the uncensored case, and let $\beta_{n'}(\theta_n)$ denote the power of the approximate α-level T_n^c test based on n' observations in the randomly censored data. Consider $n' = h(n)$ such that $\lim \beta_n(\theta_n) = \lim \beta_{n'}(\theta_n)$ where the limiting value is between 0 and 1, and let $R = \lim n/n'$. The value of $1-R$ can be viewed as a measure of the efficiency loss due to censoring. The value of R is adapted from Pittman's measure of asymptotic relative efficiency but the interpretation of R must be modified because the tests based on T_n^* and T_n^c are not competing tests in the randomly censored data (T_n^* cannot be applied to the data arising in the randomly censored model). Roughly speaking, for large n and BTR alternatives close to the null hypothesis of exponentiality, the T_n^c test requires n/R observations from the randomly censored model to do as well as the T_n^* test applied to n observations from the uncensored model. Since T_n^c and T_n^* have the same asymptotic means, it can be shown that R reduces to the limiting ratio of the null asymptotic variance of $n^{1/2}T_n^*$ to that of $n^{1/2}T_n^c$ namely,

$$R \overset{\text{def}}{=} e_H(T_n^c, T_n^*)$$

$$= \frac{(1/3) - p + p^2}{\int_0^1 [(1-p)(1-2p)I_{[0,p]}(t) - B(t)]^2 (1-t)^{-1}[L(F^{-1}(t))]^{-1} dt}$$

In order to provide a reference point to the amount of censoring, and thereby facilitate the interpretation of $e_H(T_n^c, T_n^*)$, we also include in ⟨Table 1⟩ the value of $P_H = P(X < T) = (1+\theta)^{-1}$, the probability of obtaining an uncensored observation when X is exponential with scale parameter 1 and T is independent of X and has the censoring distribution H.

3. Conclusion

In this section we have calculated efficiency loss due to censoring $e_{B}(T_n^C, T_n^*)$ for several choices with parameter p and θ. Direct calculations of $e_{B}(T_n^C, T_n^*)$ is as follows.

$$e_{B}(T_n^C, T_n^*)$$

$$= \frac{[(1/3)-p+p^2]}{[[4(3-\theta)^{-1}-4(2-p)](2-\theta)^{-1}+[(2-p)^2-(1-p)^{1-\theta}(p^2-3p+3)(1-\theta)^{-1}]]}$$

〈 Table 1 〉 $e_{B}(T_n^C, T_n^*)$ when the censoring distribution is exponential with mean $\frac{1}{\theta}$

p \ θ	1/10	1/4	1/3	1/2	2/3	3/4
0.1	0.648815	0.499587	0.420108	0.273928	0.151924	0.102039
0.3	0.391128	0.299362	0.251875	0.165802	0.093764	0.063771
0.5	0.342224	0.261962	0.220864	0.146732	0.084353	0.057994
0.7	0.583557	0.468691	0.406654	0.287596	0.177397	0.126704
0.9	0.891533	0.735026	0.661245	0.518869	0.381473	0.311594
Prob. of an uncensored obs. $=(1+\theta)^{-1}$	0.909090	0.800000	0.750000	0.666666	0.600000	0.571428

〈Table 1〉 shows that as θ decreases, the value of $e_{B}(T_n^C, T_n^*)$ increases. Note that $\theta=0$ implies no censoring. The table also shows that when p increases to 0.5, the efficiency of T_n with respect to T_n^* decreases and that when p increases from 0.5 to 1, the efficiency increases.

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