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ON POWERS OF RADICALS

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In a recent paper [1] Yu-lee Lee defined the square $\mathcal{J}^2$ of the Jacobson radical $\mathcal{J}$ and proved that $\mathcal{J}^2$ is again a radical class. It is the purpose of this brief note to prove that such an assertion is true *for all powers of all radicals* and to determine explicitly the powers of radicals.

It is well known that the class ι of all idempotent rings is a radical class. Let ρ be a radical class and

$$\rho^k = \{A|(\rho(A))^k = A\}$$

for $k = 2, 3, \cdots$.

Theorem. *For any radical class ρ the class ρ^k is a radical class, and it holds*

$$\rho^k = \rho^2 = \rho \cap \iota.$$

Proof. $(\rho(A))^k = A$ implies also $\rho(A) = A$, and so $\rho^k \subseteq \rho^2 \subseteq \rho$. A straightforward reasoning (or a verbatim quotation from the proof of [1]) shows that the class ρ^k is homomorphically closed. Thus, if $A \in \rho^k$ then also $A/A^2 \in \rho^k$ and so

$$A/A^2 = (\rho(A/A^2))^k \subseteq (A/A^2)^k = 0.$$

Hence ρ^k consists of idempotent rings which proves that $\rho^k \subseteq \rho \cap \iota$. Conversely, if $A \in \rho \cap \iota$ then $(\rho(A))^k = A^k = A$, whence $\rho \cap \iota \subseteq \rho^k$ for each $k = 2, 3, \cdots$.

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Since the intersection of radical classes is again a radical class, the proof is complete.

Notice that the Theorem holds true also for not necessarily associative rings (with the usual interpretation of the powers).

A few consequences of the Theorem:

1) *If* $\iota \subseteq \rho$ *then* $\rho^2 = \iota$, *and if* $\rho \subseteq \iota$ *then* $\rho^2 = \rho$.

2) *If* ρ *contains a non-simiprime idempotent ring, then* ρ^2 *is not hereditary,* because $A \in \rho \cap i = \rho^2$ and A contains an ideal $I \neq 0$ with $I^2 = 0$, whence $I \notin \rho^2$.

3) *If* ρ *and* τ *are radical classes and* A *is an idempotent ring and* $A \in \tau \backslash \rho$ *then* $\rho^2 \neq \tau^2$. In particular, since a simple idempotent Jacobson radical ring is not locally nilpotent (cf. [2] Proposition 22.5 and Example 32.7), and since there exist simple primitive rings without unity element, for the Levitzki radical class $\mathcal{L}$ of all locally nilpotent rings and for the Brown-McCoy radical class $\mathcal{G}$ the relation

$$\mathcal{L}^2 \subsetneq \mathcal{J}^2 \subsetneq \mathcal{G}^2$$

holds where $\mathcal{J}$ denotes the Jacobson radical class.

References

[1] Yu-lee Lee, *On squares of Jacobson radicals*, Kyungpook Math. J., 32(1992), 217-218.

[2] F. A. Szász, *Radicals of rings*, John Wiley & Sons, 1981.

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