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A NOTE ON SEMI-GROUP RINGS WHICH ARE PRE-p-RINGS

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In this note we introduce the notion of pre-p-rings to semi-group rings and obtain a necessary and sufficient condition under which a semi-group ring is a pre-p-ring. In order to obtain this we define a new class of semi-groups called pre-p-semigroups. In [1] the authors call an associative and a commutative ring R whose characteristic is p to be a pre-p-ring if $x^p y = xy^p$ for every x, y in R.

Definition. A commutative semi-group S is called a pre-p-semigroup if $x^p y = xy^p$ for all x and y in S and for a fixed prime p.

We need the following lemmas to prove our main theorem.

Lemma 1. *Let R be a pre-p-ring and S a pre-p-semigroup then the semi-group ring RS is a pre-p-ring.*

Proof. Clearly since R and S are pre-p-ring and pre-p-semigroup respectively the semi-group ring RS is commutative and associative. To prove in RS we have $x^p y = xy^p$ for every x and y in RS. Let

$$x = \sum_{i=1}^{n} x_i s_i \text{ and } y = \sum_{j=1}^{m} y_j t_j$$

with $x_i, y_j \in R$ and $s_i, t_j \in S$ $1 \leq i \leq n$ and $1 \leq j \leq m$. To prove $x^p y = xy^p$.

Consider

$$x^p y = (\sum_{i=1}^{n} x_i s_i)^p (\sum_{j=1}^{m} y_j t_j)$$

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$$= \{\sum_{i=1}^{n}(x_i s_i)^p + p(\sum \text{ (terms in products of } x_i's \; s_i's)\} \sum_{j=1}^{m} y_j t_j.$$

Since R is a pre-p-ring its characteristic is p, hence the second term in the bracket is zero.

So

$$\begin{aligned} x^p y &= (\sum_{i=1}^{n}(x_i s_i))^p (\sum_{j=1}^{m} y_j t_j) \\ &= (\sum_{i=1}^{n}(x_i^p s_i^p))(\sum_{j=1}^{m} y_j t_j) \\ &= \sum_{i,j=1}^{n,m} x_i^p y_j s_i^p t_j. \end{aligned}$$

Since R is a pre-p-ring we have

$$x_i^p y_j = x_i y_j^p \text{ for every } x_i, y_j \in R$$

and since S is a pre-p-semi-group.

We have $s_i^p t_j = s_i t_j^p$ for every s_i, t_j in S. So

$$x^p y = \sum_{i,j=1}^{n,m} (x_i y_j)^p s_i t_j^p. \qquad (I)$$

Consider $xy^p = \sum_{i=1}^{n} x_i s_i (\sum_{j=1}^{m} y_j t_j)^p$.

As before by similar reasoning we get

$$xy^p = \sum_{i,j=1}^{n,m} x_i y_j^p s_i t_j^p. \qquad (II)$$

So $x^p y = xy^p$ from (I) and (II) for every x and y in RS. Futher $px = 0$ for every x in RS. Hence RS is a pre-p-ring.

Lemma 2. *Let R be a ring with identity and S a semi-group with identity. If RS is a pre-p-ring then R is a pre-p-ring and S is a pre-p-semi-group.*

Proof. Since RS is a pre-p-ring we have $x^p y = xy^p$ and $px = 0$ for every x and y in RS. Further $R \cdot 1 \subseteq RS$. So R is evidently a pre-p-ring. To prove S is a pre-p-semi-group. We have $1 \cdot S \subseteq RS$, so S is commutative. Given in RS we have $x^p y = xy^p$ for every x, y in RS, so we have for s and t in S, $1 \cdot s, 1 \cdot t \in RS$. $(1 \cdot s)^p 1 \cdot t = (1 \cdot s)(1 \cdot t)^p$ for all s and t in S. Hence S is a pre-semi-group.

Remark. If we relax the condition, R is any ring without identity then in particular we can have R to be a commutative ring of characteristic p such that $x^p = 0$ for every x in R then RS is a pre-p-ring what ever be S.

Theorem 3. *The semi-group ring RS is a pre-p-ring if and only if R is a pre-p-ring with identity and S a pre-p-semigroup with identity.*

It is interesting to know for what groups G the group ring RG is a pre-p-ring.

Corollary. *The group ring RG is a pre-p-ring if and only if R is a pre-p-ring with identity and G is a commutative torsion group in which the order of every element in G is a divisor of $p - 1$.*

Proof. For $g \in G$ $g^p e = ge^p$ because G is a pre-p-semi-group. Thus $g^p = g$. Hence $g^{p-1} = e$. Thus (order of $g)/p - 1$. Conversely for all g, $h \in G$, $g^{p-1} = h^{p-1} = e$. Thus $g^p h = gh^p$. Thus RG is a pre-p-semi-group.

References

[1] Alexander Abian and William A. Mcworter, *On the structure of pre-p-rings*, Amer. Math. Monthly, 1964 Vol.71, 155-157.

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