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ON CERTAIN SUBCLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS

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Let $B_{n,p}(\alpha)$ be the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k z^k \ (p \in N = \{1, 2, ...\})$$

which are regular in the punctured disk $E = \{z : 0 < |z| < 1\}$ and satisfy

$$Re\{z^{p+1}(D^n f(z))'\} < -p\frac{n+\alpha}{n+1} \ (n \in N_0 = \{0, 1, 2, ...\}, |z| < 1, 0 \leq \alpha < 1),$$

where

$$D^n f(z) = \frac{1}{z^p} + \sum_{m=1}^{\infty} (p+m)^n a_{m-1} z^{m-1}.$$

It is proved that $B_{n+1,p}(\alpha) \subset B_{n,p}(\alpha)$. We also consider integrals of functions in the class $B_{n,p}(\alpha)$.

1. Introduction

Let $\sum_p$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k z^k \ (p \in N = \{1, 2, ...\}) \tag{1.1}$$

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which are regular in the punctured disk $E = \{z : 0 < |z| < 1\}$. Following Uralegaddi and Somanatha [4], we define

$$\begin{aligned} (1.2)\quad D^0 f(z) &= f(z), \\ (1.3)\quad D^1 f(z) &= \frac{1}{z^p} + (p+1)a_0 + (p+2)a_1 z + (p+3)a_2 z^2 + \cdots \\ &= \frac{(z^{p+1} f(z))'}{z^p}, \\ (1.4)\quad D^2 f(z) &= D(D^1 f(z)), \end{aligned}$$

and for $n = 1, 2, \cdots$,

$$(1.5)\qquad \begin{aligned} D^n f(z) &= D(D^{n-1} f(z)) \\ &= \frac{1}{z^p} + \sum_{m=1}^{\infty} (p+m)^n a_{m-1} z^{m-1} \\ &= \frac{(z^{p+1} D^{n-1} f(z))'}{z^p}. \end{aligned}$$

Let $B_{n,p}(\alpha)$ denote the class of functions $f(z) \in \sum_p$ which satisfy the condition

$$(1.6)\qquad Re\{z^{p+1}(D^n f(z))'\} < -p\frac{n+\alpha}{n+1} \quad (0 \leq \alpha < 1, |z| < 1).$$

Another classes defined by using above differential operator D^n have studied by Cho and Owa [1] and Uralegaddi and Somanatha [4].

In this paper, we shall show that $B_{n+1,p}(\alpha) \subset B_{n,p}(\alpha)$. Since $B_{0,p}(\alpha)$ is the class of meromorphically p-valent functions [3], all functions in $B_{n,p}(\alpha)$ are p-valent. Further properties preserving integrals are considered.

2. Some results

In proving our main results, we shall need the following lemma due to Jack [2].

Lemma 1. *Let w be non-constant regular in $U = \{z : |z| < 1\}$, $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, we have $z_0 w'(z_0) = k w(z_0)$, where k is a real number, $k \geq 1$.*

Theorem 1. *Let $f(z) \in B_{n+1,p}(\alpha)$. Then $f(z) \in B_{n,p}(\beta)$, where*

$$(2.1)\qquad \beta = \frac{n + 4 + 2\alpha(n+1)}{3(n+2)}.$$

Proof. Let $f(z) \in B_{n+1,p}(\alpha)$. Then

$$(2.2) \qquad Re\{z^{p+1}(D^{n+1}f(z))'\} < -p\frac{n+1+\alpha}{n+2}.$$

We have to show that (2.2) implies the inequality

$$(2.3) \qquad Re\{z^{p+1}(D^{n}f(z))'\} < -p\frac{n+\beta}{n+1},$$

where β is given by (2.1). Define $w(z)$ in U by

$$(2.4) \qquad z^{p+1}(D^{n}f(z))' = -p\Big\{\frac{n+\beta}{n+1} + \frac{(1-\beta)(1-w(z))}{(n+1)(1+w(z))}\Big\}.$$

Clearly, $w(z)$ is regular and $w(0) = 0$. Using the identity

$$(2.5) \qquad z(D^{n}f(z))' = D^{n+1}f(z) - (p+1)D^{n}f(z),$$

the equation (2.4) may be written as

$$(2.6) \qquad \begin{aligned} & z^{p}(D^{n+1}f(z) - (p+1)D^{n}f(z)) \\ = & -p\Big\{\frac{n+\beta}{n+1} + \frac{(1-\beta)(1-w(z))}{(n+1)(1+w(z))}\Big\}. \end{aligned}$$

Differentiating (2.6), we obtain

$$(2.7) \qquad \begin{aligned} & z^{p+1}(D^{n+1}f(z))' \\ = & -p\Big\{\frac{n+\beta}{n+1} + \frac{(1-\beta)(1-w(z))}{(n+1)(1+w(z))}\Big\} + \frac{2p(1-\beta)zw'(z)}{(n+1)(1+w(z))^2}. \end{aligned}$$

We claim that $|w(z)| < 1$ in U. For otherwise, by Lemma 1, there exists z_0 in U such that

$$(2.8) \qquad z_0 w'(z_0) = kw(z_0),$$

where $|w(z_0)| = 1$ and $k \geq 1$. Writing $w(z_0) = u + iv$, the equation (2.7) in conjuction with (2.8) yields

$$(2.9) \qquad \begin{aligned} & Re\Big\{z_0^{p+1}(D^{n+1}f(z_0))' + p\frac{n+1+\alpha}{n+2}\Big\} \\ = & \; p\Big(\frac{n+1+\alpha}{n+2} - \frac{n+\beta}{n+1}\Big) + 2p(1-\beta)kRe\Big\{\frac{u+iv}{(n+1)(1+u+iv)^2}\Big\} \\ = & \; p\Big(\frac{n+1+\alpha}{n+2} - \frac{n+\beta}{n+1}\Big) + \frac{p(1-\beta)k}{(n+1)(1+u)} \\ \geq & \; p\Big(\frac{n+1+\alpha}{n+2} - \frac{n+\beta}{n+1}\Big) + \frac{p(1-\beta)}{2(n+1)} = 0, \end{aligned}$$

since β is a root of the equation

$$3(n+2)x-(n+4+2\alpha(n+1))=0. \tag{2.10}$$

This contradicts assumption (2.2), so the proof is completed.

Since $\beta-\alpha>0$ in Theorem 1, we have the following

Corollary 1. *$B_{n+1,p}(\alpha) \subset B_{n,p}(\alpha)$ for every $n \in N_0$.*

Theorem 2. *Let $f(z) \in B_{n,p}(\alpha)$ and let*

$$F_c(z)=\frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt \ \ (c>0). \tag{2.11}$$

Then $F_c(z) \in B_{n,p}(\beta)$, where

$$\beta=\frac{1+2\alpha c}{1+2c}. \tag{2.12}$$

Proof. Let $f(z) \in B_{n,p}(\alpha)$. Define $w(z)$ in U by

$$z^{p+1}(D^nF_c(z))'=-p\Big\{\frac{n+\beta}{n+1}+\frac{(1-\beta)(1-w(z))}{(n+1)(1+w(z))}\Big\}. \tag{2.13}$$

where β is defined by (2.12). Then $w(z)$ is regular and $w(0)=0$. Using the identity

$$z(D^nF_c(z))'=cD^nf(z)-(c+p)D^nF_c(z), \tag{2.14}$$

the equation (2.13) may be written as

$$\begin{aligned} & z^p(cD^nf(z)-(c+p)D^nF_c(z)) \\ = & -p\Big\{\frac{n+\beta}{n+1}+\frac{(1-\beta)(1-w(z))}{(n+1)(1+w(z))}\Big\}. \end{aligned} \tag{2.15}$$

Differentiating (2.15), we have

$$\begin{aligned} & z^{p+1}(D^nf(z))' \\ = & -p\Big\{\frac{n+\beta}{n+1}+\frac{(1-\beta)(1-w(z))}{(n+1)(1+w(z))}\Big\}+\frac{2p(1-\beta)zw'(z)}{c(n+1)(1+w(z))^2}. \end{aligned} \tag{2.16}$$

We claim that $|w(z)|<1$ in U. For otherwise, Lemma 1, there exists z_0 in U such that $z_0w'(z_0)=kw(z_0)$, where $|w(z_0)|=1$ and $k \geq 1$. The remaining part for the proof of Theorem 2 is similar to that of Theorem 1, so we omit it.

Similarly, from Theorem 2, we have

Corollary 2. *Let* $f(z) \in B_{n,p}(\alpha)$. *Then* $F_c(z)$ *defined by* (2.11) *belongs to the class* $B_{n,p}(\alpha)$.

Using the similar method of Theorem 2, we can prove the following

Theorem 3. *Let* $f(z) \in \sum_p$ *and satisfy the condition*

$$Re\{z^{p+1}(D^n f(z))' < -p\frac{n+\alpha}{n+1} + \frac{p(1-\alpha)}{2c(n+1)} \quad (z \in U), \tag{2.17}$$

where $0 \leq \alpha < 1$ *and* $c > 0$. *Then* $F_c(z)$ *defined by* (2.11) *belongs to the class* $B_{n,p}(\alpha)$.

In case $c = 1$, Corollary 2 can be improved as follows.

Theorem 4. $f(z) \in B_{n,p}(\alpha)$ *if and only if* $F_1(z)$ *defined by* (2.11) *with* $c = 1$ *belongs to the class* $B_{n+1,p}(\alpha)$.

Proof. For the function $F_1(z)$ defined by (2.11) with $c = 1$, we have

$$z(D^n F_1(z))' + (p+1)D^n F_1(z) = D^n f(z). \tag{2.18}$$

By using the identity (2.5), the equation (2.18) reduces to

$$D^n f(z) = D^{n+1} F_1(z), \tag{2.19}$$

and the result follows.

We now prove the converse of Corollary 2.

Theorem 5. *Let* $F_c(z) \in B_{n,p}(\alpha)$ *and let* $f(z)$ *be defined as* (2.11). *Then* $f(z) \in B_{n,p}(\alpha)$ *in* $|z| < R_c$, *where*

$$R_c = \frac{\sqrt{1+c^2}-1}{c}. \tag{2.20}$$

Proof. Since $F_c(z) \in B_{n,p}(\alpha)$, we can write

$$-z^{p+1}(D^n F_c(z))' = p(\frac{n+\alpha}{n+1} + \frac{(1-\alpha)u(z)}{n+1}), \tag{2.21}$$

where $u(z) \in P$, the class of functions with positive real part in U and normalized by $u(0) = 1$. Using the equation (2.14) and differentiating (2.21), we obtain

$$-\frac{z^{p+1}(D^n f(z))' + p\frac{n+\alpha}{n+1}}{\frac{p(1-\alpha)}{n+1}} = u(z) + \frac{zu'(z)}{c}. \tag{2.22}$$

Using the well known estimate, $\frac{|zu'(z)|}{Reu(z)} \le \frac{2r}{1-r^2}(|z| = r)$, the equation (2.22) yields

$$Re\Big\{-\frac{z^{p+1}(D^n f(z))' + p\frac{n+\alpha}{n+1}}{\frac{p(1-\alpha)}{n+1}}\Big\} \ge Reu(z)\Big\{1 - \frac{2r}{c(1-r^2)}\Big\}. \tag{2.23}$$

Now the right hand side of (2.23) is positive provided $r < R_c$. Hence $f(z) \in B_{n,p}(\alpha)$ for $|z| < R_c$.

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