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# ON OSCILLATIONS OF NEUTRAL PARABOLIC DIFFERENTIAL EQUATIONS

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## 1. Introduction

Recently the fundamental theory for partial differential equations with deviating arguments has undergone an intensive development. However, the equalitative theory of these important classes for applications of partial differential equations is still in a initial stage of its development. Especially, only a few results have been published so far which deal with the oscillatory properties of the solutions of parabolic differential equations with deviating arguments. We refer the reader to the papers by Bykov and Kultaev [1], Kreith and Ladas [4], Yoshid [7], [8], Mishev and Bainov [5], and Cui [2]. But the necessary and sufficient condition for oscillation of solutions of parabolic equations has not been studied.

In the present necessary and sufficient conditions for oscillations of the solutions of neutral linear parabolic equations of the form

$$\begin{aligned}&\frac{\partial}{\partial t}[u(x,t)+\sum_i a_i u(x+\tau_i)]+bu(x,t)+\sum_j b_j u(x,t+\sigma_j)\\ =&\ \triangle u(x,t)+\sum_k c_k \triangle u(x,t+\rho_k),\quad (x,t)\in\Omega\times(0,\infty)\equiv G,\quad (1)\end{aligned}$$

are obtained, where $\Omega$ is a bounded domain in $\mathbf{R}^n$ with a piecewise smooth boundary, $\triangle u(x,t)=\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}u(x,t)$, $I,J,K$ are initial segments of natural numbers and $a_i,\tau_i,b,\ b_j,\sigma_j,c_k,\rho_k\in\mathbf{R}$ for $i\in I, j\in J$ and $k\in K$.

Consider boundary conditions of the form

$$\frac{\partial u}{\partial n}=0,\quad (x,t)\in\partial\Omega\times[0,\infty),\qquad (2)$$

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$$u = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \tag{3}$$

**Definition 1**. The classical solution $u(x,t)$ of (1) satisfying the boundary condition (2) or (3) is oscillatory on $G$ if $u(x,t)$ has a zero on $\Omega \times [t,\infty)$ for any $t > 0$.

## 2. Main results

In the following theorems a necessary and sufficient conditon for oscillation of the solutions of problems (1), (2) and (1), (3) in the domain $G$ is obtianed.

With each solution $u(x,t)$ of problem (1), (2) we associate the function

$$v(t) = \int_\Omega u(x,t)dx, \quad t \geq 0, \tag{4}$$

**Lemma 1**. *Let $u(x,t)$ be a solution of problem (1), (2). Then the function $v(t)$ defined by (4) satisfies the differential equation*

$$\frac{d}{dt}[v(t) + \sum_i a_i v(t+\tau_i)] + bv(t) + \sum_j b_j v(t+\sigma_j) = 0, \quad t \geq t_0, \tag{5}$$

*where $t_0$ is sufficiently large positive number.*

*Proof.* Let $u(x,t)$ be a solution of problem (1), (2). Introduce the notation

$$t_0 = \max\{|\tau_i|, |\sigma_j|, |\rho_k| : i \in I, j \in J, k \in K\}.$$

Integrating both sides of (1) with respect to $x$ over the domain $\Omega$, for $t \geq t_0$ we obtain

$$\frac{d}{dt}[\int_\Omega u(x,t)dx + \sum_i a_i u(x,t+\tau_i)dx] + b\int_\Omega u(x,t)dx$$

$$+ \sum_j b_j \int_\Omega u(x,t+\sigma_j)dx - [\int_\Omega \Delta u(x,t)dx \tag{6}$$

$$+ \sum_k c_k \int_\Omega \Delta u(x,t+\rho_k)dx] = 0. \tag{7}$$

From Green's formula it follows that

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} ds = 0, \tag{8}$$

$$\int_\Omega \Delta u(x,t+\rho_k)dx = \int_{\partial\Omega} \frac{\partial u}{\partial n}(x,t+\rho_k)ds = 0, \quad k \in K. \tag{9}$$

Using (7) and (8), from (6) we obtain that

$$\frac{d}{dt}[v(t) + \sum_i a_i v(t+\tau_i)] + bv(t) + \sum_j b_j v(t+\sigma_j) = 0.$$

This completes the proof of the lemma.

In the domain $\Omega$ consider the Dirichlet problem

$$\Delta U(x) + \alpha U(x) = 0, \quad x \in \Omega, \tag{10}$$

$$U(x) = 0, \quad x \in \partial\Omega, \tag{11}$$

where $\alpha =$ const. It is well known [6] that the smallest eigenvalue $\alpha_0$ of problem (9), (10) is positive and the corresponding eigenfunction $\varphi(x)$ can be chosen so that $\varphi(x) > 0$ for $x \in \Omega$.

With each solution $u(x,t)$ of the problem (1), (3) we associate the function

$$w(t) = \int_\Omega u(x,t)\varphi(x)dx, \quad t \geq 0. \tag{12}$$

**Lemma 2**. *Let $u(x,t)$ be a solution of problem (1), (3). Then the function $w(t)$ defined by (11) satisfies the differential equation*

$$\begin{aligned} &\frac{d}{dt}[w(t) + \sum_i a_i w(t+\tau_i)] + bw(t) + \sum_j b_j w(t+\sigma_j) \\ + \quad &\alpha_0[w(t) + \sum_k c_k w(t+\rho_k)] = 0, \quad t \geq t_0, \end{aligned} \tag{13}$$

*where $t_0$ is a sufficiently large positive number.*

*Proof.* Let $u(x,t)$ be a solution of the problem (1), (3). Introduce the notation

$$t_0 = \max\{|\tau_i|, |\sigma_j|, |\rho_k| : i \in I, j \in J, k \in K\}.$$

Multiplying both sides of (1) by the eigenfunction $\varphi(x)$ and integrating with respect to $x$ over the domain $\Omega$, for $t \geq t_0$ we obtain

$$\begin{aligned} &\frac{d}{dt}[\int_\Omega u(x,t)\varphi(x)dx + \sum_i a_i \int_\Omega u(x,t+\tau_i)\varphi(x)dx] \\ + \quad &b\int_\Omega u(x,t)\varphi(x)dx + \sum_j b_j \int_\Omega u(x,t+\sigma_j)\varphi(x)dx \\ - \quad &[\int_\Omega \Delta u(x,t)\varphi(x)dx + \sum_k c_k \int_\Omega \Delta u(x,t+\rho_k)\varphi(x)dx] = 0. \end{aligned} \tag{14}$$

From Green's formula it follows that

$$\begin{aligned}\int_\Omega \Delta u(x,t)\varphi(x)dx &= \int_\Omega u(x,t)\Delta\varphi(x)dx \\ &= -\alpha_0\int_\Omega u(x,t)\varphi(x)dx = -\alpha_0 w(t), \qquad (15)\end{aligned}$$

$$\begin{aligned}\int_\Omega \Delta u(x,t+\rho_k)\varphi(x)dx &= \int_\Omega u(x,t+\rho_k)\Delta\varphi(x)dx \\ &= -\alpha_0\int_\Omega u(x,t+\rho_k)\varphi(x)dx \qquad (16) \\ &= -\alpha_0 w(t+\rho_k).\end{aligned}$$

Using (14) and (15), from (13) we obtain that

$$\begin{aligned}&\frac{d}{dt}[w(t)+\sum_i a_i w(t+\tau_i)] + bw(t) + \sum_j b_j w(t+\sigma_j) \\ &+\alpha_0[w(t)+\sum_k C_k w(t+\rho_k)] = 0.\end{aligned}$$

This completes the proof of the lemma.

From the above lemmas it follows that it suffices to investigate conditions for oscillation of the solutions of (1) in the domain $\Omega$ to fund conditions for the oscillatory properties of neutral ordinary differential equations of the form

$$\frac{d}{dt}[x(t)+\sum_i p_i x(t+\tau_i)] + \sum_j q_j x(t+\sigma_j) = 0, \quad t \geq t_0, \qquad (17)$$

where $I$ and $K$ are initial segments of natural numbers and $p_i, \tau_i, q_j, \sigma_j \in \mathbf{R}$ for $i \in I$ and $j \in J$.

**Definition 2.** The solution $x(t)$ of the differential equation (16) is called oscillatory if it has arbitrarily large zeros. Otherwise it is called nonoscillatory.

In the proof of the subsequent theorems we shall use the following result of [3].

**Theorem 1** (Grammatikopoulos and Stavroulakis [3]). *A necessary and sufficient condition for the oscillation of all solutions of (16) is that is characteristic equation*

$$\lambda + \lambda\sum_i p_i e^{\lambda\tau_i} + \sum_j q_j e^{\lambda\sigma_j} = 0. \qquad (18)$$

*has no real roots.*

A corollary of Lemma 1 and Theorem 1 is the following necessary and sufficient condition for oscillation of the solutions of problem (1), (2).

**Theorem 2.** *A necessary and sufficient condition for all solutions of problem (1), (2) to oscillate in the domain G is that the equation*

$$\lambda + \lambda \sum_i a_i e^{\lambda \tau_i} + \sum_j b_j e^{\lambda \sigma_j} = 0. \tag{19}$$

*has no real roots.*

*Proof.* (Necessity) If $\lambda_0 \in \mathbf{R}$ is a root of (18), then the function $u(x,t) = e^{\lambda_0 t}$ is a nonoscillating positive solution of problem (1), (2) in the domain $\Omega \times [t_0, \infty)$.

(Sufficiency) Suppose that the assertion is not true and let $u(x,t)$ be a nonoscillation solution of problem (1), (2). Let $u(x,t) > 0$ for $(x,t) \in G$. (The case when $u(x,t) < 0$ for $(x,t) \in G$ is considered analogously).

From Lemma 1 it follows that the function $v(t)$ defined by (4) is a positive solution of the differential equation (5). Then from Theorem 1 applied to (5) it follows that (18) has a real root, which contradicts the condition of the theorem.

A corollary of Lemma 2 and Theorem 1 is the following necessary and sufficient condition for oscillation of the solutions of problem (1), (3).

**Theorem 3.** *A necessary and sufficient condition for all solutions of problem (1), (3) to oscillate in the domain G is that the equation*

$$\lambda[1 + \sum_i a_i e^{\lambda \tau_i}] + b + \sum_j b_j e^{\lambda \sigma_j} + \alpha_0[1 + \sum_k c_k e^{\lambda \rho_k} = 0 \tag{20}$$

*has no real roots.*

*Proof.* (Necessity) If $\lambda_0 \in \mathbf{R}$ is a root of (19), then the function $u(x,t) = e^{\lambda_0 t}\varphi(x)$ is a nonoscillating positive solution of problem (1), (3) in the domain $\Omega \times [t_0, \infty)$.

(Sufficiency) Suppose that the assertion is not true and let $u(x,t)$ be a nonoscillating solution of problem (1), (3). Let $u(x,t) > 0$ for $(x,t) \in G$ (The case when $u(x,t) < 0$ for $(x,t) \in G$ is considered analogously). From Lemma 2 it follows that the function $w(t)$ defined by (11) is a positive solution of the differential equation (12). Then from Theorem 1 applied to (12) it follows that (19) has a real root, which contradicts the condition of the theorem.

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