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A REMARK ON STRONGLY NONLINEAR PARABOLIC PROBLEMS

A. T. El-Dessouky

In the present paper we deal with the existence of weak solutions of strongly nonlinear variational inequalities of parabolic operators of the form

$$\frac{\partial u(x,t)}{\partial t} + Au(x,t), \quad (x,t) \in Q = \Omega \times (0,T) \tag{1}$$

$$u(0) = 0;$$

where $Au(x,t) = \sum_{i=1}^{n} D_i A_i(x,t,D_i u(x,t) + A_0(x,t,u(x,t))$, is strongly nonlinear in the sense that its coefficients has a libral growth. Although we restrict ourselves to second order operators, our results are still workable for higher order operators.

Introduction

Consider the parabolic initial-boundary value problem

$$\frac{\partial u}{\partial t} + A(u) + g(x,t,u) = f \text{ in } Q$$

$$u(0) = 0 \text{ in } \Omega$$

$$D^{\alpha} u|_{0,T)\times\partial\Omega} = 0, |\alpha| \le m-1$$

where

$$A(u)(x,t) = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x,t,Du),$$

D^{α} denotes the partial derivatives with respect to the space variables corresponding to the multi-indices $\alpha = (\alpha_1, \cdots, \alpha_N)$ of order $|\alpha| = \sum_{i=1}^{N} \alpha_i$, Du stands for the function u with all its derivatives up to order m.

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If the coefficients $A_\alpha(x,t,\xi)$ are subject to some growth condition implying $A(u)$ to be a bounded operator of a Sobolev space, respectively Sobolev-Orlicz space, into its dual, while $g(x,t,u)=0$; the existence of weak solutions is obtained by the theory of monotone operators. Brezis and Browder [2] proved that for $g(x,t,u)$ the growth condition can be substituted by certain structure conditions. Landes [5] extended the above result of [2] to include the top order coefficients. In [6] he discussed sufficient conditions for the perturbed term to establish the existence of weak solutions. In [3] Browder and Brezis generalized their results [2] to the class of variational inequalities. It is the purpose of this work to extend the results of Landes [5],[7] to the corresponding class of variational inequalities of arbitrary order.

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Prerequisites

Let Ω be a bounded domain in $\mathbf{R}^N$ having a smooth boundary $\partial\Omega$. Let us introduce for $A(u)$ the following hypothesis.

A1) For $i=0,1,\cdots,N$, each $A_i(x,t,\xi)$ is continuous in t and ξ for almost all x and measurable in x for all t and ξ with $A_i(x,t,0)=0$ for all $(x,t)\in Q$. Moreover, for any $s\geq 0$ there exists a function $h_s\in L^1_+(Q)$ such that for all $(x,t)\in Q$.

$$\sup_{|r|\leq s}|A_i(x,t,r)|\leq h_s(x,t)$$

A2) For all $(x,t)\in Q$ and $r\in\mathbf{R}$,

$$\sum_{i=1}^{N}A_i(x,t,r)r\geq c_0|r|^p-k_1(x,t),\quad A_0(x,t,r)r\geq 0$$

with $c_0>0$ and $k_1\in L^1(Q)$.

A3) For all $(x,t)\in Q$ and $r,r^*\in\mathbf{R}$,

$$\sum_{i=1}^{N}(A_i(x,t,r)-A_i(x,t,r^*))(r-r^*)>0.$$

For the Galerkin method we choose a sequence $\Psi_j\subset C_0^\infty(\Omega)$ such that $\cup_{n=1}^\infty W_n$ with $W_n=span(\Psi_1,\cdots,\Psi_n)$ is dense in $W^{j,p}(\Omega), j>\frac{N}{P}+1$.

Denote by $Y_n = c(0,T;W_n)$. Since the closure of $\cup Y_n$ with respect to the norm

$$\|u\|_{\substack{1,0\\C(Q)}} := \sup_{\substack{i=1,2,\cdots,N\\(x,t)\in Q}} |D_i u(x,t)|$$

contains $C_0^\infty(Q)$, we have that for f there exists $f_k \in \cup_{n=1}^\infty Y_n$ such that $f_k \to f$ in $L^2(Q)$ [7]. Let K be a closed convex subset of $C(0,T;L^2(\Omega))$ with $0 \in K$. Introduce the space

$$X = L^2(Q) \cap L^1(0,T;W_0^{1,1}(\Omega))$$

with the norm

$$\|\cdot\|_x = \|u\|_2 + \|u\|_{1,1,1}$$

where $\|\cdot\|_2$ is the usual norm in $L^2(Q)$ and $\|\cdot\|_{1,1,1}$ is defined by

$$\|u\|_{1,1,1} = \int_Q \sum_{i=1}^N |D_i u| dx dt.$$

For more details of these spaces we refer to [1].

Definition 1. A function $u_n \in Y_n \cap K (n \in \mathbf{N})$ is called a Galerkin solution of the variational inequalities of the operator (1) if and only if

$$\int_\Omega (\frac{\partial u_n}{\partial t})(v-u_n)dx + < Tu_n(t), v-u_n(t) > \geq \int_\Omega f_n(t)(v-u_n)dx$$

for all $t \in [0,T]$ and all $v \in Y_n \cap K$,

$$u_n(0) = 0,$$

where

$$< Tu_n(t), v-u_n(t) > = \int_\Omega \sum_{i=0}^N A_i(t, D_i u_n(t))(D_i v - D_i u_n(t))dx.$$

The existence of a Galerkin solution and its main property which is a consequence of our hypotheses is stated in the following lemma.

Lemma. *For every $n \in \mathbf{N}$ there exists a Galerkin solution $u_n \in Y_n \cap K$ such that*

$$\|u_n\| \leq c_1 \quad (c_1 > 0). \tag{2}$$

Proof. Define a vector-valued function $g_n(t,a) : [0,T] \times \mathbf{R}^N \to \mathbf{R}^N$ by

$$(g_n(t,a))_j = \inf_{\Omega} \sum_{i=1}^{N} A_i(\sum_{k=1}^{N} D_k\Psi_k)D_i\Psi_j, a = (a_1,\cdots,a_N).$$

Clearly $g_n(t,a)$ is continuous. Therefore the system of ordinary differential inequalities

$$(\xi(t) + g_n(t,\xi), n - \xi(t)) \geq (f_n(t), \eta - \xi(t)) \text{ for a.a. } t \in [0,T]$$
$$\xi(0) = 0 \tag{3}$$

with $(f_n(t))_i = \int_\Omega f_n(t)\psi_i dx$, has a local solution [8].

From (3) we get the estimate

$$\frac{1}{2}\frac{d}{dt}|\xi(t)| \leq |f_n(t)||\xi(t)|,$$

and hance,

$$|\xi(t)| \leq C_n(T)$$

including the existence of a solution $\xi_n \in Y_n \cap K$. In exactly the same manner as in [5] we may prove (2).

Definition 2. A function $u \in X \cap K$ is called a weak solution of (1) if

$$(\frac{\partial u}{\partial t}, v-u) + (Tu, v-u) \geq (f, v-u) \text{ for all } v \in C^1(0,T;C_0^\infty(\Omega)) \cap K$$

with

$$u(0) = 0$$

where

$$(Tu,w) = \int_\Omega \sum_{i=0}^{N} A_i(x,t,D_iu)D_iw dxdt.$$

Existence Theorem

Theorem. *Let K be a closed convex subset of $C(0,T;L^2(\Omega))$ containing the origin. Let the hypotheses A1) - A3) be satisfied. Then for a given $f \in C^1(0,T;L^2(\Omega))$ the problem (1) admits a weak solution.*

Proof. From the above lemma we obtain the existence of a Galerkin solution u_n such that

$$(\dot{u}_n, v-u_n) + (Tu_n, v-u_n) \geq (f_n, v-u_n), \quad f_n, v \in Y_n \cap K, \tag{4}$$

and

$$f_n \to f \text{ in } L^2(Q).$$

Setting $v = 0$ in (4) we obtain the uniform boundedness from above the numerical sequence $\{A_i(x,t,D_iu_n)D_iu_n\}_{i=0}^N$. The proof will follow if we show the following assertions for some subsequence of (u_n):

$$\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \text{ weakly in } L^2(Q) \tag{5}$$

$$u_n \to u \text{ strongly in } L^P(Q) \tag{6}$$

$$\sum_{i=0}^N A_i(x,t;D_iu_n)D_iw \to \sum_{i=0}^N A_i(x,t;D_iu)D_iw \quad \forall w \in C_0^\infty(Q) \tag{7}$$

$$\liminf \sum_{i=0}^N A_i(x,t;D_iu_n)D_iu_n \geq \sum_{i=0}^N A_i(x,t;D_iu)D_iu \quad \forall n \tag{8}$$

$$u \in X \cap K. \tag{9}$$

Assertions (6) - (9) follows in exactly the same manner as in [4] and [5]. The crucial point to prove is to show assertion (5). In order to prove (5). Given $\rho > 0$, an $n \in \mathbf{N}$ and any $w_n \in K \cap Y_n$; put $v = u_n - \rho w_n$. Since v is arbitrary, w_n is absolutely arbitrary for a given u_n. Substituting in (4) we get

$$< \dot{u}_n(t), w_n(t) > + < Tu_n(t), w_n(t) > \leq < f_n(t), w_n(t) > .$$

In particular, since v is arbitrary we have from (4):

$$- < \dot{u}_n(t), \frac{u_n(t-\rho) - u_n(t)}{-\rho} > - < Tu_n(t), \frac{u_n(t-\rho) - u_n(t)}{-\rho} >$$
$$\leq \; - < f_n(t), \frac{u_n(t-\rho) - u_n(t)}{-\rho} > .$$

Allowing $\rho \to 0$, we get

$$- < \dot{u}_n(t), \dot{u}_n(t) > - < Tu_n(t), \dot{u}_n(t) > \leq - < f_n(t), \dot{u}_n(t) > . \tag{10}$$

Similarly, from (4), we have,

$$< \dot{u}_n(t+\rho), \frac{u_n(t+\rho) - u_n(t)}{\rho} > + < Tu_n(t+\rho), \frac{u_n(t+\rho) - u_n(t)}{\rho} >$$
$$\leq \; < f_n(t+\rho), \frac{u_n(t+\rho) - u_n(t)}{\rho} > .$$

Let $\rho \to 0$, we get

$$\lim_{\rho\to 0}\{< \dot{u}_n(t+\rho), \dot{u}_n(t) > + < Tu_n(t+\rho), \dot{u}_n(t) > \leq \lim_{\rho\to 0} < f_n(t+\rho), \dot{u}_n(t) > . \tag{11}$$

Adding (10) and (11) and integration over $(0,t)$, we obtain

$$\begin{aligned} &\lim_{\rho\to 0}\int_0^t\int_\Omega [\frac{\dot{u}_n(s+\rho)-\dot{u}_n(s)}{\rho}]\dot{u}_n(s)dsdx \\ &+\lim_{\rho\to 0}\frac{1}{\rho^2}\int_0^t\int_\Omega (Tu_n(s+\rho)-Tu_n(s))(u_n(s+\rho)-u_n(s))dsdx \\ \leq\ &\lim_{\rho\to 0}\int_0^t\int_\Omega [\frac{f_n(t+\rho)-f_n(s)}{\rho}]\dot{u}_n(s)dsdx. \end{aligned}$$

Using A2), A3), we get

$$\int_\Omega |\dot{u}_n(t)|^2 - |\dot{u}_n(0)|^2 dx \leq 2\int_0^t\int_\Omega |f_n(s)-\dot{u}_n(s)|^2 dxds.$$

Therefore,

$$\|\dot{u}_n(t)\|^2_{L^2(\Omega)} \leq 2\int_0^t \|f'n(s)\|^2_{L^2(\Omega)}\|\dot{u}_n(s)\|^2_{L^2(\Omega)}ds + \|\dot{u}_n(0)\|^2_{L^2(\Omega)} \tag{12}$$

On the other hand, we get from (4),

$$< \dot{u}_n(t), \dot{u}_n(t) > + < Tu_n(t), \dot{u}_n(t) > \leq < f_n(t), \dot{u}_n(t) > .$$

In particular, in view of A1) we have

$$< \dot{u}_n(0), \dot{u}_n(0) > \leq < f_n(0), \dot{u}_n(0) >$$

$$\|\dot{u}_n(0)\|^2_{L^2(\Omega)} \leq \frac{1}{2}\|f'n(0)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\dot{u}_n(0)\|^2_{L^2(\Omega)}. \tag{13}$$

Now from (12) and (13) as well as Gronwall's inequality we conclude the a priori bound

$$\|\dot{u}_n(0)\|^2_{L^2(\Omega)} \leq \text{ const.} \quad \forall n \in \mathbf{N}.$$

and assertion (5) follows.

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Department of Mathematics, Faculty of Science, Helwan University, Cairo, Egypt.