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STABLE SOLUTIONS OF SCALAR DIFFERENTIAL DELAY EQUATIONS

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1. Introduction

Consider the equation

$$(\mu, f) \qquad \dot{x}(t) = -\mu x(t) + f(x(t-1))$$

for $\mu \geq 0$ and $f : \mathbf{R} \to \mathbf{R}$ a continuous function. This is the simplest differential equation for a system governed by decay and delayed feedback. In case

$$\text{(NF)} \qquad \xi f(\xi) < 0 \qquad \text{for all real} \quad \xi \neq 0$$

feedback is negative. Equation (μ, f) has applications in ecology, physiology and physics. See for example the references in [6]. Systems of equations of this type are used to describe neuronal nets [7].

Initial data $\phi : [-1, 0] \to \mathbf{R}$ in the infinite-dimensional phase space $C = C([-1,0], \mathbf{R})$ define continuous solutions $x^{\phi} : [-1, \infty) \to \mathbf{R}$ which satisfy (μ, f) for $t > 0$. The relations

$$F(t, \phi) = x_t^{\phi}, \qquad x_t^{\phi} = x^{\phi}(t+s) \in C, \qquad s \in [-1, 0], \quad \text{for} \quad t \geq 0$$

give rise to a semiflow F on C. A class of solutions which are fundamental for the dynamics of (μ, f) is the class of solutions which are slowly oscillating in the sense that consecutive zeros are spaced at distances

$$z_{j+1} - z_j > 1$$

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greater than the delay.

Assume from now on that

(M) f is C^1 with $f(0) = 0$ and $f'(\xi) < 0$ for all $\xi \in \mathbf{R}$.

In a recent study it is shown that condition (M) and one-sided boundedness ($\sup f < \infty$ or $\inf f > -\infty$) imply that parts of the global attractor for (μ, f) are as nice as one could wish. They are 2-dimensional graphs in the phase space C, C^1-smooth, annulus-like and bordered by the orbits of two other slowly oscillating periodic solutions (or by one such orbit, and by the stationary state $0 \in C$) [10].

In the present paper we prove that certain slowly oscillating periodic solutions of the equation (μ, f) are hyperbolic and there orbits Γ in the phase space are orbitally exponentially asymptotically stable.

2. Hyperbolic periodic orbits

In this section we consider the autonomous functional differential equation

$$\dot{x}(t) = f(x_t) \tag{1}$$

where $f : C \to \mathbf{R}$ is of class C^2. If y is a nonconstant τ-periodic solution of (1), then $\dot{y}(t) \neq 0$ for all t and the *orbit* $\Gamma = \bigcup_t y_t$ of y is a closed curve. The linear variational equation relative to y is

$$\dot{z}(t) = Df(y_t)z_t. \tag{2}$$

Since $\dot{y}(t) = f(y_t)$ for $t \in \mathbf{R}$, $\ddot{y}$ exists and

$$\ddot{y}(t) = Df(y_t)\dot{y}_t$$

and $\dot{y} \neq 0$ implies that $\lambda = 1$ is a characteristic multiplier of (2). If the multiplier $\lambda = 1$ is simple, we say that the periodic orbit Γ is *nondegenerate.*

Since equation (1) is autonomous, $y(t + \alpha)$ for any $\alpha \in \mathbf{R}$ is also a τ-periodic solution of (1) and the orbit of $y(t + \alpha)$ is also Γ. The linear variational equation relative to $y(t + \alpha)$ is

$$\dot{z}(t) = Df(y_{t+\alpha})z_t. \tag{3}$$

A nondegenerate periodic orbit Γ of equation (1) is said to be ***hyperbolic*** if 1 is the only multiplier of equation (2) with modulus equal to 1.

Suppose the τ-periodic orbit Γ is hyperbolic and let $T(t,\sigma)$ be the solution operator of the linear equation (2). Then $T(t+\alpha,\sigma+\alpha)$ for $\alpha \in \mathbf{R}$ is the solution operator of equation (3). The map $T(\alpha+\tau,\alpha)$ induces a decomposition of the space C into invariant subspaces as $C = C^u(\alpha) \oplus C^o(\alpha) \oplus C^s(\alpha)$, where $C^u(\alpha)$ corresponds to the multipliers of equation (3) with modulus greater than 1, $C^o(\alpha)$ corresponds to the simple multiplier of equation (3) which is equal to 1 with a basis $\{\dot{y}_\alpha\}$, and $C^s(\alpha)$ is a complementary subspace for which initial values in $C^s(\alpha)$ correspond to solutions of equation (3) approaching zero as $t \to \infty$.

Let $p^u : C \to C^u(0)$ and $p^s : C \to C^s(0)$ be projections onto $C^u(0)$ and $C^s(0)$ respectively. For $B_\nu = \{\phi \in C| \quad |\phi| < \nu\}$, $\nu > 0$, let

$$S(\nu,\alpha) = C^s(\alpha) \cap B_\nu \qquad U(\nu,\alpha) = C^u(\alpha) \cap B_\nu$$

and define

$$\mathcal{S}(\nu,\tau) = \bigcup_{0\le\alpha\le\tau} (y_\alpha + H(S(\nu,\alpha),\alpha))$$

$$\mathcal{U}(\nu,\tau) = \bigcup_{0\le\alpha\le\tau} (y_\alpha + G(U(\nu,\alpha),\alpha))$$

where $H(\phi^s,\alpha)$ and $G(\phi^u,\alpha)$ are continuously differentiable functions defined on $S(\nu,\alpha)$ and $U(\nu,\alpha)$ respectively. For the detail see chapter 10 in [2].

In fact, the sets $\mathcal{S}(\nu,\tau)$ and $\mathcal{U}(\nu,\tau)$ are the local stable and unstable manifolds for the orbit Γ as in the following theorem.

THEOREM 2.1. *Suppose that equation (1) has continuous first and second derivatives and Γ is a hyperbolic orbit of equation (2) generated by a periodic function y. Then there exist positive constants K,γ and ν such that the following properties are satisfied :*

(1) *$\mathcal{S}(\nu,\alpha)$ is a local stable set for Γ and, for $0<\alpha<\tau$, $\mathcal{S}(\nu,\alpha)$ is diffeomorphic to $[0,1]\times\mathcal{S}(\nu,0)$. Also, for any $\phi \in \mathcal{S}(\nu,\alpha)$, there is a constant $\alpha(\phi)$ such that the solution $x(\phi)$ of equation (1) satisfies*

$$|x(\phi)(t) - y(t+\alpha(\phi))| \le Ke^{-\gamma t}|\phi - y_{\alpha(\phi)}|, \qquad t \ge 0$$

(2) *$\mathcal{U}(\nu,\tau)$ is a local unstable for Γ and, for $0<\alpha<\tau$, $\mathcal{U}(\nu,\alpha)$ is diffeomorphic to $[0,1]\times\mathcal{U}(\nu,0)$. Also, for any $\psi \in \mathcal{U}(\nu,\alpha)$, there*

is a constant $\beta(\psi)$ *such that the solution* $x(\psi)$ *of equation (1) satisfies*

$$|x(\psi)(t) - y(t + \beta(\psi))| \le Ke^{\gamma t}|\psi - y_{\beta(\psi)}|, \qquad t \le 0.$$

(3) *There is a neighborhood* W *of* Γ *such that, for any* $\phi \in W \backslash S(\gamma, \sigma)$, *the solution* $x(\phi)$ *of equation (1) must leave* W *in finite time.*

Proof. See chapter 10 in [2].

When all characteristic multipliers of Γ except 1 is inside the unit circle, we can obtain asymptotic stability of Γ immediately from (1) and (2) of the theorem.

COROLLARY 2.2. *If* Γ *is an hyperbolic periodic orbit of equation (1) with characteristic multipliers except* 1 *inside the unit circle, then* Γ *is exponentially asymtotically sable with asymptotic phase.*

3. Symmetric periodic solutions without decay

In this section we consider the nonlinear differential delay equation

$$(f) \qquad \dot{x}(t) = f(x(t-1)).$$

Suppose that the continuous function $f : \mathbf{R} \to \mathbf{R}$ satisfies

$$(\mathrm{M}) \quad f \text{ is } C^1 \text{ with } f(0) = 0 \text{ and } f'(\xi) < 0 \text{ for all } \xi \in \mathbf{R}.$$

$$(\mathrm{O}) \qquad f(\xi) = -f(-\xi) \quad \text{for all} \quad \xi \in \mathbf{R}.$$

Note that (M) implies (NF). And assume that

$$(\mathrm{D}) \qquad f'(0) < -\frac{\pi}{2} < \lim_{\xi \to +\infty} \frac{f(\xi)}{\xi}.$$

Under these conditions, it is known that (f) has slowly oscillating symmetric periodic solutions i.e., solutions $y : \mathbf{R} \to \mathbf{R}$ which have minimal period $\tau = 4$ and share the symmetry

$$(\mathrm{S}) \qquad y(t) = -y(t-2) \quad \text{for all} \quad t \in \mathbf{R}.$$

The zeros of y form a sequence $\{z_j | j = 0, 1, \cdots, J, J \in \mathbf{N} \text{ or } J = \infty\}$ such that

$$z_j + 1 < z_{j+1} \quad \text{and} \quad y'(z_{j+1}) \neq 0 \quad \text{for} \quad j < J.$$

By translation in time, we may assume

$$z_o(y) = -1 \quad \text{and} \quad 0 < y(t) \quad \text{for} \quad -1 < t \leq 0.$$

Using simplicity of zeros and condition (M), one sees that y increases on [-1,0] from 0 to $y(0)$, with $0 < y'(s)$ for $-1 \leq s < 0$, $y' < 0$ on (0,1].

The condition (S) yields

$$z_j = 2j - 1, \qquad j \in \mathbf{Z}$$

and one concludes that

$$y(\mathbf{R}) = [-y(0), y(0)].$$

See [1], [4], and [5].

The Floquet multipliers of a periodic solution y of equation (f) are given by the linear variational equation along y

$$\dot{v}(t) = f'(y(t-1))v(t-1).$$

Differentiation of equation (f) shows that $\lambda = 1$ is a multiplier, with eigenvector $\dot{y}_0$ of the monodromy operator. In case, $m(1) = 1$, y is called ***nondegenerate***, and y is said to be ***hyperbolic*** if 1 is the only multiplier with modulus 1.

THEOREM 3.1. *Suppose f satisfies (M) and (O). Assume that*

$$\text{(D)} \qquad f'(0) < -\frac{\pi}{2} < \lim_{\xi \to +\infty} \frac{f(\xi)}{\xi}$$

and

$$\text{(DI)} \qquad f' \quad \textit{is increasing on} \quad [0, \infty).$$

Then every symmetric periodic solution y of equation (f) is nondegenerate with $|\lambda| < 1$ for all Floquet multipliers λ except the trivial one $\lambda = 1$.

Proof. See [1].

The orbit Γ of a periodic solution y of equation (f) is the set of segments $y_t \in C$, where $y_t(s) = y(t+s)$ for $s \in [-1, 0]$ and $t \in \mathbf{R}$.

THEOREM 3.2. *In addition to the hypotheses of Theorem 3.1, suppose that f is of class C^2. Then the orbit Γ of y is exponentially asymptotically stable with asymptotic phase.*

Proof. It follows immediately from Corollary 2.2 and Theorem 3.1.

4. Periodic solutions for small decay

The results in section 3 can be extended to the case of small μ and to functions which are close to an f satisfying the assumptions of Theorem 3.2.

For $\mathcal{U} \subset C$ open, let $C_b^1(\mathcal{U}, \mathbf{R})$ be the space of bounded C^1-mappings from $\mathcal{U}$ to$\mathbf{R}$ with bounded derivative. This space is equipped with the C^1-norm

$$|g| := \max\{\sup_{\psi\in\mathcal{U}}|g(\psi)|, \sup_{\psi\in\mathcal{U}}|Dg(\psi)|\}.$$

For $g \in C_b^1(\mathcal{U}, \mathbf{R})$, consider the functional differential delay equation initial value problems

$$(\psi, g) = \begin{cases} x_0 = \psi \in \mathcal{U} \\ \dot{x}(t) = g(x_t), \quad t \geq t_0, \end{cases}$$

where $x_t \in C$ is a segment of x. The maximal solutions $x^{(\psi,g)}$ of these IVPs exist on interval of possibly finite length. They define a local semiflow on $\mathcal{U}$:

$$\Phi(\cdot,\cdot,g) : \mathbf{R} \times \mathcal{U} \to C$$
$$(t,\psi) \mapsto \Phi(t,\psi,g) := x_t^{(\psi,g)}.$$

Assume now $f \in C_b^1(\mathcal{U}, \mathbf{R})$, and that

(UC) $\quad Df$ is uniformly continuous.

Assume further that $\phi \in \mathcal{U}$, and $y : \mathbf{R} \to \mathbf{R}$ is a periodic solution of (ϕ, f) with minimal period $\tau_0 > 1$. This implies that the orbit $\Gamma(y)$ of y is contained in $\mathcal{U}$, and that y is a periodic trajectory of the semiflow $\Phi(\cdot,\cdot,f)$. Assume now that $H \subset C$ is a closed hyperplane i.e, $H = h^{-1}\{c\}$ for $h \in C^*$ and $c \in \mathbf{R}$, and that with $\phi = y_0$

(1) $h(y_0) = c$
(2) $h(\dot{y}_0) \neq 0$

Condition (2) means that the tangent vector

$$D_1\Phi(\tau_0,\phi,f)1=\dot{y}_0=\dot{\phi}$$

to the closed orbit $\Gamma(x)$ at ϕ is not contained in the tangent space to H, i.e., the phase curve is transversal to H at ϕ. Under these assumptions, we can define the Poincaré-mapping P associated with y, H and the period τ_0, on a neighborhood $\phi+U_0$ of ϕ in H, where U_0 is a neighborhood of $0\in H_0:=h^{-1}\{0\}$. P sends initial values from this neighborhood to the points where their trajectories hit H again after a time which is approximately τ_0. ϕ is a fixed point of P, and the mapping $P_0:U_0\ni\phi\mapsto P(\phi+\psi)-\phi\in H_0$ is C^1.

Assume now that y is a hyperbolic periodic trajectory in the sense that ϕ is a hyperbolic fixed point of P_0 i.e., the linear operator $DP_0(0)\in L(H_0,H_0)$ has no spectrum on the unit circle in C. The theorem below shows the persistence of the hyperbolic periodic solution and a continuous change of the Poincaré-mapping and its derivative with respect to g. The main ideas of the proof of the theorem are taken from [4].

THEOREM 4.1. *Under the above assumptions, there exist open balls $U_0\ni 0$ in H_0, $\mathcal{G}\ni f$ in $C_b^1(\mathcal{U},\mathbf{R})$, an open interval $I\ni\tau_0$, and a C^1-mapping*

$$\Pi:\mathcal{G}\to\phi+U_0\subset H\subset C.$$

such that the following hold:

(1) *For all $g\in\mathcal{G}$, there is a unique initial value ϕ^g in $\phi+U_0$, such that the solution x^g of (ϕ^g,g) is periodic with minimal period in I. ϕ^g is given by $\phi^g:=\Pi(g)$.*

(2) *For $g\in\mathcal{G}$, let $\Gamma(x^g)$ be the periodic orbit in C corresponding to x^g. These phase curves $\Gamma(x^g)$ are transversal to H at ϕ^g. There is a family of Poincaré-mappings $P^g(g\in\mathcal{G})$, associated with x^g,H, the period of x^g in I and the semiflow $\Phi(\cdot,\cdot,g)$. The P^g are all defined on the (fixed) set $\phi+U_0$. Define $P_0^g\in C^1(U_0,H_0),P_0^g(\psi):=P^g(\phi+\psi)-\phi$. Then,*

$$P_0^g\to P_0^f=P_0\quad\text{and}\quad DP_0^g\to DP_0^f$$

for $g\to f$ (in $C_b^1(\mathcal{U},\mathbf{R})$), uniformly on U_0.

(3) *$\phi^g\to\phi^f=\phi$ in C, and $DP_0^g(\phi^g-\phi)\to DP_0^f(0)$ in $L(H_0,H_0)$ for $g\to f$. The linear operators $DP_0^g(\phi^g-\phi)$ are hyperbolic for $g\in\mathcal{G}$. Let U_g (resp.S_g be the unstable (resp., stable) space of*

$DP_0^g(\phi^g - \phi)$, and let $p_g^u \in L(H_0, H_0)$ be the projection onto U_g determined by $H_0 = U_g \oplus S_g$. Then, $p_g^u \to p_f^u$ for $g \to f$. In particular, if U_f is finite dimensional, the dimensions of U_g and U_f coincide for $|g - f|$ small enough.

To prove the theorem we shall need following two lemmas.

LEMMA 4.2. *Let f and g be in $C_b^1(\mathcal{U}, \mathbf{R})$ with*

$$|g - f| < \varepsilon.$$

If x and y are solutions to (x_{t_0}, f) and (y_{t_0}, g), respectively, and $x_{t_0} = y_{t_0}$, then for some $K > 0$

$$|x(t) - y(t)| \le \frac{\varepsilon}{K}\exp((K|t - t_0|) - 1) \quad \text{for} \quad t \ge t_0.$$

Proof. For $t \ge t_0$, we have

$$\begin{aligned} x(t) - y(t) &= \int_{t_0}^{t} [\dot{x}(s) - \dot{y}(s)]\, ds - x(t_0) + y(t_0) \\ &= \int_{t_0}^{t} [f(x_s) - g(y_s)]\, ds \\ &= \int_{t_0}^{t} [f(x_s) - f(y_s) + f(y_s) - g(y_s)]\, ds. \end{aligned}$$

Since f is locally Lipschitz function, there exists $K_1 > 0$ such that $|f(x_s) - f(y_s)| \le K_1|x_s - y_s|$. Hence

$$\begin{aligned} |x(t) - y(t)| &\le \int_{t_0}^{t} \{|f(x_s) - f(y_s)| + |f(y_s) - g(y_s)|\}\, ds \\ &\le \int_{t_0}^{t} \{K_1|x_s - y_s| + \varepsilon\}\, ds. \end{aligned}$$

Since $|x(s) - y(s)|$ is uniformly continuous on $[t_0, t]$, there exists $K_2 > 0$ such that

$$|x_s - y_s| \le K_2|x(s) - y(s)| \quad \text{for} \quad t_0 \le s \le t.$$

Therefore

$$|x(t) - y(t)| \le \int_{t_0}^{t} \{K_1K_2|x(s) - y(s)| + \varepsilon\}\, ds$$

and setting $K = K_1K_2$, we have

$$|x(t) - y(t)| + \frac{\varepsilon}{K} \le \frac{\varepsilon}{K} + K \int_{t_0}^{t} \{|x(s) - y(s)| + \frac{\varepsilon}{K}\}\, ds.$$

Using Gronwall's inequality, we obtain the desired inequality.

This lemma implies that, if $|f - g|$ is uniformly small, solutions of $\dot{x}(t) = f(x_t)$ and $\dot{y}(t) = g(y_t)$ having the same initial value, are to be close.

Lemma 4.3. *Let U and $\mathcal{B}$ be open subsets of C and $C_b^1(\mathcal{U}, \mathbf{R})$, respectively. Then the mapping $\Phi : [0,T] \times U \times \mathcal{B} \to C; (t, \phi, f) \mapsto \Phi(t, \phi, f)$ is C^1.*

Proof.

(1) Since $D_1\Phi(t, \varphi, f)$ is just $f(y_t)$, it is continuous.

(2) For small $\psi \in \mathcal{U}$, $D_2\Phi(t, \phi, f) \in L(\mathcal{U}, C)$ is the linear map $\psi \mapsto v_t^{(\psi,f)}$ where $v_t^{(\psi,f)}$ is the solution of linear variational equation $\dot{v}(t) = Df(\Phi(t, \phi, f))v_t$. In fact, using Gronwall's inequality, we can obtain

$$\lim_{\psi \to 0} \frac{|\Phi(t, \phi + \psi, f) - \Phi(t, \phi, f) - v_t^{(\psi,f)}|}{|\psi|} = 0.$$

And by continuity of solutions in initial conditions and data to the linear variational equation, $D_2\Phi(t, \phi, f)$ is continuous.

(3) Consider the differential equation

$$(\mathcal{V}) \qquad \dot{v}^{(\phi,g)}(t) = Df(y_t^{(\phi,f)})(v_t^{(\phi,g)} - \phi(t_0)) - g(y_t^{(\phi,f)}).$$

Then the solutions of $(\mathcal{V})$ define the linear map from $\mathcal{B}$ into C which maps

$g \mapsto v_t^{(\phi,g)}$. Since $y^{(\phi,f)}$ is continuous on $[t_0-1,t]$,

$$\begin{aligned}
&|y^{(\phi,g+f)}(t) - y^{(\phi,f)}(t) - (v^{(\phi,g)}(t) - \phi(t_0))| \\
&=|\int_{t_0}^{t} (g+f)(y_s^{(\phi,g+f)})\,ds - \int_{t_0}^{t} f(y_s^{(\phi,f)})\,ds \\
&\quad - \int_{t_0}^{t} \{Df(y_s^{(\phi,f)})(v_s^{(\phi,g)} - \phi(t_0)) + g(y_s^{(\phi,f)})\}\,ds| \\
&\le \int_{t_0}^{t} |f(y_s^{(\phi,g+f)}) - f(y_s^{(\phi,f)}) - Df(y_s^{(\phi,f)})(v_s^{(\phi,g)} - \phi(t_0)) \\
&\quad + (g(y_s^{(\phi,g+f)}) - g(y_s^{(\phi,f)}))|\,ds \\
&= \int_{t_0}^{t} |Df(y_s^{(\phi,f)})\{(y_s^{(\phi,g+f)} - y_s^{(\phi,f)}) - (v_s^{(\phi,g)} - \phi(t_0))\} \\
&\quad + R_1 + Dg(y_s^{(\phi,f)})(y_s^{(\phi,g+f)} - y_s^{(\phi,f)}) + R_2|\,ds \\
&\le \int_{t_0}^{t} K_0|Df(y_s^{(\phi,f)})||y^{(\phi,g+f)}(s) - y^{(\phi,f)}(s) - (v^{(\phi,g)}(s) - \phi(t_0))|\,ds \\
&\quad + \int_{t_0}^{t} |R_1 + Dg(y_s^{(\phi,f)})(y_s^{(\phi,g+f)} - y_s^{(\phi,f)}) + R_2|\,ds
\end{aligned}$$

for some $K_0 > 0$, where R_1 and R_2 are the remainders in Taylor expansion. For every $\varepsilon > 0$, there exist positive constants K_1, K_2 which satisfy $|R_1| < \varepsilon K_1|g|$, $|R_2| < \varepsilon K_2|g|$. Let

$$u(t) = |y^{(\phi,g+f)}(t) - y^{(\phi,f)}(t) - (v^{(\phi,g)}(t) - \phi(t_0))|$$

and $N = |f(y_t^{(\phi,f)})|$, then

$$u(t) \le N\int_{t_0}^{t} u(s)\,ds + K\varepsilon.$$

By Gronwall's inequality, we obtain

$$\lim_{g\to 0} \frac{|\Phi(t,\phi,g+f) - \Phi(t,\phi,f) - (v_t^{(\phi,g)} - \phi(t_0))|}{|g|} = 0.$$

And since by Lemma 4.1 and (UC), the mapping $g \mapsto v_t^{(\phi,g)}$ is continuous in $\mathcal{B}$, $D_3\Phi = v_t^{(\phi,g)} - \phi(t_0)$ is continuous.

Proof of Theorem 4.1. By Lemma 4.2 and Lemma 4.3, there exist $T > \tau_0 > 1$, balls U around ϕ in C and $\mathcal{B}$ around f in $C_b^1(\mathcal{U}, \mathbf{R})$ such that for every $g \in \mathcal{B}$, $[0,T] \times U$ is contained in the domain of definition of the local semi flow $\Phi(\cdot,\cdot,g)$. We have $\Phi(\tau_0,\phi,f) = \phi \in H$ so that $h(\Phi(\tau_0,\phi,f)) = c$ and $h(D_1\Phi(\tau_0,\phi,f)1) = h(\dot{y}_0) \neq 0$. Using Implicit Function Theorem, we obtain open balls $U_1 \subset U$ centered at $\phi \in C$, $\mathcal{B}_1 \subset \mathcal{B}$ centered at f in $C_b^1(\mathcal{U}, \mathbf{R})$ and an open interval $I \subset (1,T)$ containing τ_0 with a C^1 function $\rho : U_1 \times \mathcal{B}_1 \to I$ satisfying the following properties:

(a) $\rho(\phi, f) = \tau_0$
(b) for all $(\psi, g) \in U_1 \times \mathcal{B}_1$, $\Phi(\rho(\psi,g),\psi,g) \in H$.

$\rho(\psi, g)$ is the usual return time, which depends not only on the initial value but also on the function g of the FDE (ψ, g).

Moreover, for $g \in \mathcal{B}_1$, we can obtain the associated Poincaré-type mapping

$$\tilde{P}^g : U_1 \to C, \quad \tilde{P}^g(\psi) := \Phi(\rho(\psi,g),\psi,g).$$

Since ρ and Φ are C^1, $\tilde{P}^g \in C_b^1(U_1, C)$ for $g \in \mathcal{B}_1$. Futhere, by Lemma 4.3, the mapping

$$Q : \mathcal{B}_1 \ni g \mapsto \tilde{P}^g \in C_b^1(U_1, C)$$

is continuous at f. Here $\tilde{P}^g$ have to be restricted to $U_1 \cap H$ to get the P^g.

In fact, consider the C^1-mapping

$$\begin{aligned} \mathcal{F} : \quad \mathcal{B}_1 \times H_0 \quad &\longrightarrow \quad H_0, \\ (g,\psi) \quad &\mapsto \quad P^g(\phi+\psi) - (\phi+\psi) = \Phi(\rho(\phi+\psi,g),\phi+\psi,g) - (\phi+\psi) \\ &= P_0^g(\psi) - \psi. \end{aligned}$$

The assumptions of Theorem 4.1 imply that

(a) $\mathcal{F}(f,0) = 0$, since ϕ is a fixed point of P^f,
(b) $D_2 f(f,0) = DP_0^f(0) - id_{H_0}$ is an isomorphism of H_0, since $DP^f(0)$ is hyperbolic and there, in particular, its spectrum does not contain 1.

Now, the Implicit Function Theorem yields a function $g \mapsto \Psi(g)$ defined for g in a subset $\mathcal{G} \ni f$ of $\mathcal{B}_1$ with $\mathcal{F}(g, \Psi(g)) = 0$, so that $\Psi(g)$ is a fixed point of P_0^g. By continuity, $DP_0^g(\Psi(g))$ will be hyperbolic for $|g - f|$ small. Finally, the initial value $\Pi(g) := \psi + \Psi(g)$ is a fixed point of P^g, so it has a periodic trajectory under $\Phi(\cdot,\cdot,g)$. Now the convergence results of the theorem follow from the continuity of the mappings Q and Π.

The assertion about the p_g^u follows from

$$p_g^u = id_{H_0} - \frac{1}{2\pi i}\int_\gamma (\lambda - DP^g(\varphi^g - \phi))^{-1}\, d\lambda,$$

where $\gamma; [0, 2\pi] \to \mathbf{C}, \gamma(t) = e^{it}$.

Applying Theorem 4.1 and by the results in section 3, we obtain the stable slowly oscillating hyperbolic periodic solutions to some scalar differential delay equations (μ, f) with small decay for restricted class of nonlinear functions.

In the following theorem, for $g \in C^1(\mathbf{R}, \mathbf{R})$ and for some fixed $R > 0$, we put the norm

$$|g|_{C^1} := \max\{\max_{t\in[-R,R]} |g(t)|, \max_{t\in[-R,R]} |g'(t)|\}.$$

THEOREM 4.4. *Suppose that $f : \mathbf{R} \to \mathbf{R}$ satisfies*

(M) f *is* C^1 *with* $f(0) = 0$ *and* $f'(\xi) < 0$ *for all* $\xi \in \mathbf{R}$

(O) $$f(\xi) = -f(-\xi) \quad \text{for all} \quad \xi \in \mathbf{R}$$

(D) $$f'(0) < -\frac{\pi}{2} < \lim_{\xi\to+\infty} \frac{f(\xi)}{\xi}$$

(DI) f' *is increasing on* $[0, \infty)$

Then

(1) *there exist an open ball $U \ni y_0$ in C and $\hat{\mu}$, $\gamma > 0$ such that for all $\mu \in \mathbf{R}$ and for all $g \in C^1(\mathbf{R}, \mathbf{R})$ with*

$$|\mu| < \hat{\mu} \quad \text{and} \quad |f - g|_{C^1} < \gamma,$$

there is a unique initial value $\varphi(\mu, g)$ in U, which defines a periodic solution of

(μ, g) $$\dot{z}(t) = \mu z(t) + g(z(t-1)),$$

and this periodic solution is hyperbolic and stable,

(2) *if g is of class C^2, then the orbit of x is exponentially stable with the asymptotic phase.*

Proof. (1) From the preceding statements of and with Theorem 3.1, there is a nondegenerate hyperbolic periodic solution y with period greater than 1 of equation (f). The orbit of y in C is contained in $\mathcal{U}$ and transversal to the hyperplane $H = \{\psi \in C | \psi(-1) = 0\}$.

Take $R > 0$ with $y(\mathbf{R}) \subset (-R, R)$ and let $\mathcal{U} := C(R)$, the open ball with center at 0 and of radius R in C.

Define a mapping $\hat{f} : \mathcal{U} \to \mathbf{R}$ by

$$\hat{f}(\psi) := f(\psi(-1)).$$

Then $\hat{f} \in C_b^1(\mathcal{U}, \mathbf{R})$ and y satisfies $\dot{y}(t) = \hat{f}(y_t)$, $t \in \mathbf{R}$.

Since for $\phi \in \mathcal{U}$, $\psi \in C$,

$$D\hat{f}(\phi) = Df(\phi(-1))\psi(-1)$$

and Df is uniformly continuous on $[-R, R]$, $D\hat{f}$ satisfies (UC).

For $g \in C^1(\mathbf{R}, \mathbf{R})$ and $\mu \in \mathbf{R}$, the mapping

$$\hat{g}_\mu(\psi) := \mu\psi(0) + g(\psi(-1)) \quad \text{for} \quad \psi \in \mathcal{U}$$

is an element of $C_b^1(\mathcal{U}, \mathbf{R})$ and depends only on μ and g. And the decay delay defferential equation (μ, g) is equivalent to the functional differential equation $\dot{z}(t) = \hat{g}_\mu(z_t)$. Note that $\hat{f} = \hat{f}_0$ and y satisfies the conditions of Theorem4.1. Therefore it suffices to prove that if $\mu \to 0$ and $|f - g|_{C^1} \to 0$ then

$$\hat{g}_\mu \to \hat{f} \quad \text{in} \quad C_b^1(\mathcal{U}, \mathbf{R}).$$

This is immediate, since for $\psi \in \mathcal{U}$

$$\begin{aligned} |\hat{g}_\mu(\psi) - \hat{f}(\psi)| \le & |\mu| R + \sup_{t \in [-R, R]} |f(t) - g(t)| \\ \le & |\mu| R + |f - g|_{C^1}, \end{aligned}$$

and since for $\phi \in \mathcal{U}$, $\psi \in C$

$$\begin{aligned} |D\hat{g}_\mu(\phi)\psi - \hat{f}(\phi)\psi| = & |\mu\psi(0) + Dg(\phi(-1))\psi(-1) - Df(\phi(-1))\psi(-1)| \\ \le & (|\mu| + |f - g|_{C^1})|\psi|. \end{aligned}$$

(2) The result follows immediately from Theorem 3.2 and Theorem 4.1.

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