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ON THE STRUCTURE OF CLOSED IDEALS IN GROUP ALGEBRAS

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0. Introduction

Let G be a locally compact abelian group with the dual group Γ, and $L^1(G)$ be the (Banach) group algebra of G, consisting of all integrable functions on G with respect to Haar measure dx. The Fourier transform of a function f in $L^1(G)$ is defined on Γ by

$$\hat{f}(\gamma) = \int_G f(x)\overline{(x,\gamma)}dx \quad (\gamma \in \Gamma),$$

and let $A(\Gamma)$ denote the set of all functions $\hat{f}$ so obtained. Then Γ can be identified with character space of $L^1(G)$ and itself is a locally compact group with the weak topology induced by $A(\Gamma)$.

It is, in general, important but unsolved to completely describe all the closed ideals in an arbitrary commutative Banach algebra although for some specific Banach algebras ([7], [19]) simple descriptions of the closed ideals may be available. In this paper we mainly deal with $L^1(G)$ of a locally compact abelian group G and obtain some classifications and properties of closed ideals in $L^1(G)$.

Let $f \in L^1(G)$. The zero set of f is defined to be $Z(f) = \{\gamma \in \Gamma : \hat{f}(\gamma) = 0\}$. For an ideal I in $L^1(G)$, the hull of I is $h(I) = \cap\{Z(f) : f \in I\}$.

In §2 the condition that a closed ideal I in $L^1(G)$ has a bounded approximate identity is characterized in various way. In particular, we show that I has a bounded approximate identity if and only if $h(I) \in \Sigma(\Gamma)$, where $\Sigma(\Gamma)$ is the coset ring of Γ.

The aim of §3 is to improve the results of H. Helson in [8]. We also obtain chains of ideals of $L^1(G)$ by showing that there is an ideal of $L^1(G)$ strictly between any two ideals $I \subsetneq J$ which have the same hull whenever either I or J is closed.

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1. Preliminaries

Let G be a locally compact group. A closed subalgebra B of $L^1(G)$ is said to have a bounded left approximate identity if there is a net $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ in B such that $\lim\|u_\alpha * f - f\| = 0$ for all $f \in B$ and $\sup\|u_\alpha\| < \infty$. Right and two-sided approximate identities are defined analogously, and by an approximate identity we mean it is two-sided, otherwise stated.

It is noted that B has a bounded left approximate identity if and only if there is a $k > 0$ such that for any $f_1, f_2, \cdots, f_n \in B$ and $\varepsilon > 0$ there exists a $u \in B$ with $\|u\| \le k$ and $\|u * f_i - f_i\| \le \varepsilon$ $(i = 1, 2, \cdots, n)$.

The following lemma is due to Cohen [2].

LEMMA 1.1. *Let G be a locally compact group and suppose that $L^1(G)$ has a left approximate identity. Then for each $f \in L^1(G)$ and $\delta > 0$, there exist functions g and h in $L^1(G)$ such that*

(1) $f = g * h$
(2) $\|f - g\| < \delta$
(3) *g belongs to the closed left ideal generated by f.*

DEFINITION 1.2. Let G be a locally compact group. A left invariant mean m is a positive linear functional on $L^\infty(G)$ such that $m(1_G) = 1$ and $m(f_x) = m(f)$ for all $f \in L^\infty(G)$ and $x \in G$, where $f_x(y) = f(xy)$ $(y \in G)$.

A locally compact group G is said to be amenable if there exists a left invariant mean m on $L^\infty(G)$.

LEMMA 1.3. *A locally compact group G is amenable if and only if for any compact set $K \subset G$ and $\varepsilon > 0$ there exists an $h \in L^1(G)$ such that $h \ge 0$, $\|h\| = 1$ and $\|h_x - h\| \le \varepsilon$ for all $x \in K$.*

Proof. See [15, chapter 8].

Let $\mathfrak{L}(L^1(G))$ denote the space of all continuous linear mappings from $L^1(G)$ to itself. For every $T \in \mathfrak{L}(L^1(G))$ and $h \in L^1(G)$, define

$$T_h f = \int_G h(y)(T(f_y))_{y^{-1}}dy \qquad (f \in L^1(G)).$$

Then clearly $T_h \in \mathfrak{L}(L^1(G))$ and $\|T_h\| \le \|h\|\|T\|$. It also follows from straightfoward computation that

$$\|f * T_h g - T_h(f * g)\| \le \|g\|\|T\| \int_G |f(x)|\|h_x - h\|dx \cdots\cdots (1.1)$$

LEMMA 1.4. *Let G be a locally compact group. If G is amenable, then for $f_1, f_2, \cdots, f_n \in L^1(G)$ and $\varepsilon > 0$ there exists an $h \in L^1(G)$ such that $h \geq 0$, $\|h\| = 1$ and for each $i = 1, 2, \cdots, n$,*

$$\|f_i * T_h g - T_h(f_i * g)\| \leq \varepsilon \|g\| \|T\| \qquad (g \in L^1(G),\ T \in \mathfrak{L}(L^1(G))).$$

Proof. Let $f_1, f_2, \cdots, f_n \in L^1(G)$ and $\varepsilon > 0$ be given. For each $i = 1, 2, \cdots, n$, we may assume $\|f_i\| \leq 1$. Take a compact set $K \subset G$ such that

$$\int_{G \setminus K} | f_i(x) | \, dx \leq \frac{1}{4}\varepsilon \qquad (i = 1, 2, \cdots, n).$$

Then since G is amenable, by Lemma 1.3, there is an $h \in L^1(G)$ such that $h \geq 0$, $\|h\| = 1$ and $\|h - h_x\| \leq \frac{1}{4}\varepsilon$ for all $x \in K$. Now, for every $i = 1, 2, \cdots, n$,

$$\int_K | f_i(x) | \, \|h - h_x\| dx \leq \int_K | f_i(x) | \, dx \cdot \frac{1}{4}\varepsilon = \|f_i\| \cdot \frac{1}{4}\varepsilon \leq \frac{1}{4}\varepsilon,$$

and

$$\int_{G \setminus K} | f_i(x) | \, \|h - h_x\| dx \leq \int_{G \setminus K} | f_i(x) | \, dx \cdot 2\|h\| \leq \frac{1}{2}\varepsilon,$$

so the result follows from (1.1).

2. Approximate identities and closed ideals

It is known that if G is compact then, for a closed ideal I in $L^1(G)$, there exists a continuous surjective projection $P: L^1(G) \longrightarrow I$ such that $f * Pg = P(f * g)$ for every $f, g \in L^1(G)$ [18, Theorem 1]. Further, if in addition G is abelian, then there exists an idempotent measure μ on G such that $Pf = f * \mu$. In this case P is actually an algebra homomorphism and it follows that I, regarded as a Banach algebra, has a bounded approximate identity. We first in this section find out what happen if G is not compact or abelian.

PROPOSITION 2.1. *Let G be a locally compact group and I a closed left ideal in $L^1(G)$. If G is amenable and there exists a continuous onto projection $P: L^1(G) \longrightarrow I$, then I has a right approximate identity bounded by $\|P\|$.*

Proof. For every $f \in L^1(G)$ we have $P(f_x)_{x^{-1}} \in I$ for all $x \in G$. Hence

$$P_h f = \int_G h(x)(Pf_x)_{x^{-1}} dx$$

is well-defined, and thus every P_h maps $L^1(G)$ into I. Moreover, if $f \in I$ then $f_x \in I$ for every $x \in G$ and so $(Pf_x)_{x^{-1}} = f_{xx^{-1}} = f$. Thus if $\int_G h = 1$ then P_h is surjective.

We now take $f_1, f_2, \cdots, f_n \in I$ and $\varepsilon > 0$. Then by Lemma 1.4 and the above there exists a projection E of $L^1(G)$ onto I such that $\|E\| \le \|P\|$ and, for each $i = 1, 2, \cdots, n$,

$$\|f_i * Eg - E(f_i * g)\| \le \varepsilon \|P\|^{-1} \|E\| \|g\| \le \varepsilon \|g\|$$

for every $g \in L^1(G)$.

Choose $g \in L^1(G)$ so that $\|g\| = 1$ while $\|f_i * g - f_i\| \le \varepsilon \|E\|^{-1}$ for each i. Then $Eg \in I$, $\|Eg\| < \|P\|$, and

$$\begin{aligned}\|f_i * Eg - f_i\| &\le \|f_i * Eg - E(f_i * g)\| + \|E(f_i * g) - Ef_i\| \\ &\le 2\varepsilon \qquad (i = 1, 2, \cdots, n)\end{aligned}$$

and this completes the proof.

COROLLARY 2.2. *With the same conditions as above, a right (two-sided) closed ideal in $L^1(G)$ has a left (two-sided) bounded approximate identity.*

We remark that the condition of amenability of G in the theorem can not be dropped. If G is not amenable, $I_0(G) = \{f \in L^1(G) \colon \int_G f = 0\}$, so called the augmentation ideal, is a two-sided closed ideal such that there is a continuous projection of $L^1(G)$ onto $I_0(G)$. On the other hand, $I_0(G)$ does not have a bounded right approximate identity [23].

We now turn to the case that G is abelian. We shall show that for a closed ideal I in $L^1(G)$ the existence of a bounded approximate identity in I is equivalent to various conditions.

THEOREM 2.3. *Let G be a locally compact abelian group, and I be a closed ideal in $L^1(G)$. The following are equivalent;*

(1) *I has a bounded approximate identity,*
(2) *I is factorizable, that is, for every $f \in I$ and $\varepsilon > 0$, there exists $k > 0$ such that $f = g_1 * g_2$ for some $g_1, g_2 \in I$ with $\|g_1\| \le k$ and $\|f - g_2\| \le \varepsilon$,*
(3) *There exists a continuous projection of $L^\infty(G)$ onto $I^\perp = \{h \in L^\infty(G) \colon (f, h) = 0 \text{ for all } f \in I\}$.*

Proof. (1) $\Rightarrow$ (2) is the special case of Lemma 1.1. For (2) $\Rightarrow$ (1), let $f \in I$ and $\varepsilon > 0$. Then there exists a constant k, and we have g_1 and g_2 as in (2).

Furthermore, $\|g_1 * f - f\| = \|g_1 * (f - g_2)\| \leq k\varepsilon$.

(1) $\Rightarrow$ (3) ; Let τ be the natural map $L^1(G) \to L^\infty(G)^*$ and $\{u_\alpha\}_{\alpha \in \mathcal{A}}$ a bounded approximate identity in I. By Alaoglu theorem the net $\{\tau u_\alpha\}_{\alpha \in \mathcal{A}}$ has a w*-convergent cofinal subnet and hence we may assume that $\{\tau u_\alpha\}_{\alpha \in \mathcal{A}}$ is itself w*-convergent in $L^\infty(G)^*$. Then, for every $h \in L^\infty(G)$, there exists $\lim(u_\alpha, h)$. Consider the map $E\colon L^\infty(G) \to L^\infty(G)$ such that $(f, Eh) = (f, h) - \lim(u_\alpha, f' * h)$ for each $f \in L^1(G)$ and $h \in L^\infty(G)$, where $f'(y) = f(y^{-1})$. Then E is continuous linear.

If $f \in I$, then for all $h \in L^\infty(G)$, $(f, Eh) = (f, h) - \lim(f * u_\alpha, h) = 0$. Therefore E maps $L^\infty(G)$ onto $I^\perp$. Conversely, for any $h \in I^\perp$ and $f \in L^1(G)$, $(f, Eh - h) = \lim(u_\alpha, f' * h) = \lim(f * u_\alpha, h) = 0$, so $Eh = h$.

Thus E is a projection of $L^\infty(G)$ onto $I^\perp$.

For (3) $\Rightarrow$ (1), we note from the amenability of G (since G is abelian) that there exists an idempotent measure μ on G such that $I = L^1(G) * \mu$. Now define $Pf = f * \mu$ for all $f \in L^1(G)$. Then P is a continuous projection of $L^1(G)$ onto I, and thus the result follows from Corollary 2.2.

We now prove one of the main theorems in this section so that the redult could be added into the above theorem.

Let $\Sigma(\Gamma)$ denote the coset ring of Γ, that is, the Boolean algebra of subsets of Γ generated by the cosets of all subgroups of Γ.

For a closed set Ω in Γ, set

$$F(\Omega) = \{f \in L^1(G)\colon Z(f) \text{ is a neighborhood of } \Omega\}$$

$$H(\Omega) = \{f \in L^1(G)\colon \Omega \subset Z(f)\}$$

$$K(\Omega) = \{f \in F(\Omega)\colon \hat{f} \text{ has a compact support.}\}.$$

The set Ω is a set of <u>spectral synthesis</u> if $H(\Omega)$ is the closed ideal in $L^1(G)$ whose hull is Ω. We say that Ω is a <u>Ditkin set</u> if, for each $f \in H(\Omega)$, there is a sequence $\{f_n\} \subset F(\Omega)$ such that $\|f_n * f - f\| \to 0$ as $n \to \infty$, and Ω is a <u>strong Ditkin set</u> if the above sequence $\{f_n\}$ can be chosen independently of $f \in H(\Omega)$.

LEMMA 2.4. *Every closed set in $\Sigma(\Gamma)$ is a strong Ditkin set.*

Proof. See [20, Th. 2.6].

THEOREM 2.5. *Let G be a locally compact abelian group with dual group Γ. Then a closed ideal I in $L^1(G)$ has a bounded approximate identity if and only if $h(I) \in \Sigma(\Gamma)$.*

Proof. Suppose I has a bounded approximate identity $\{u_\alpha\}_{\alpha \in \mathcal{A}}$ and $\|u_\alpha\| < k$. Let $\bar{G}$ be the Bohr compactification of G [17, 1.8.1] and $M(\bar{G})$ denote the measure algebra of $\bar{G}$.

By the definition of hull of I, for each $\gamma \in \Gamma \setminus h(I) = h(I)'$ there exists an $f \in I$ such that $\hat{f}(\gamma) = 1$. Since $\hat{u}_\alpha(\gamma)\hat{f}(\gamma) \to \hat{f}(\gamma)$, we see that $\hat{u}_\alpha \to \chi_{h(I)'}$ pointwise on Γ. If we consider each u_α as a measure on $\bar{G}$ then as a net in the weakly compact ball of measures $\{\mu \in M(\bar{G}) \colon \|\mu\| \leq k\}$, $\{u_\alpha\}_{\alpha \in \mathcal{A}}$ must have a subnet $\{u_{\alpha\beta}\}$ converging weakly to some $\mu \in M(\bar{G})$. Now since each $\gamma \in \Gamma$ extends to a continuous function on $\bar{G}$, $\hat{u}_{\alpha\beta}(\gamma) \to \hat{\mu}(\gamma)$ pointwise on Γ. Thus $\hat{\mu}$ is the characteristic function of $h(I)'$, that is, $\hat{\mu} = \chi_{h(I)'}$, and μ is an idempotent measure on G. By Cohen's theorem on idempotent measure [17, Th.3.1.3] we conclude that $h(I)$ lies in the coset ring of Γ.

Conversely, suppose that $h(I) \in \Sigma(\Gamma)$ and let $\Omega = h(I)$. Then, by Lemma 2.4, for $f_1, f_2, \cdots, f_n \in H(\Omega)$ and $\varepsilon > 0$, there is a constant c and $u \in F(\Omega)$ such that $\|u\| \leq c$ and $\|u * f_i - f_i\| \leq \varepsilon$ $(i = 1, 2, \cdots, n)$. But if $\varepsilon > 0$ and $f_1, f_2, \cdots, f_n \in I$ are given, set $\delta = \min\{1, \varepsilon/2 \max \|f_i\|\}$ and choose u as above for $\varepsilon/2$.

Now since $L^1(G)$ has an approximate identity [11, 31E], we can take $v \in L^1(G)$ such that $\|v\| = 1$ and $\|u - v * u\| < \delta/2$.

It follows from [17, Th. 2.6.6] that there exists $w \in L^1(G)$ such that the support of $\hat{w}$ is compact and $\|v - w\| < \delta/(2\|u\| + 1)$. Thus we have
$\|u * w - u\| < \delta$,
$\|u * w\| \leq \|u\|(\|v - w\| + \|v\|) \leq 2\|u\| \leq 2c$, and for each i, $\|u * w * f_i - f_i\| \leq \|f_i\|\|u * w - u\| + \|u * f_i - u\| < \varepsilon$.

But $u * w \in K(\Omega) \subset I$, and this implies that I has an approximate identity.

By Lemma 2.4, we obtain the following as a supplementary result.

COROLLARY 2.6. *Let G be a locally compact abelian group. If a closed ideal I in $L^1(G)$ has a bounded approximate identity, then $h(I)$ is a strong Ditkin set.*

3. Chains of ideals in $L^1(G)$

We in this section develop ascending or descending chains of ideals in $L^1(G)$ by showing that, for any two ideals $I \subsetneqq J$ with $h(I) = h(J)$, there exists an ideal of $L^1(G)$ strictly between I and J whenever one of them is closed. Thus the result we shall show in Theorem 3.5 is somewhat improvement of H. Helson's result in [8] or [17, Th.7.7.2].

DEFINITION 3.1. For an ideal I in $L^1(G)$, a function $f \in L^1(G)$ is said to <u>belong to I</u> <u>locally at the point γ</u> in Γ, denoted by $f \in \tilde{I}_{(\gamma)}$, if there is an open neighborhood U of γ and some $g \in I$ such that $\hat{f}(\eta) = \hat{g}(\eta)$, $\eta \in U$, that is, $f - g \in F(\{\gamma\})$.

PROPOSITION 3.2. *Let G be a locally compact abelian group with dual group Γ, and let $I \subsetneqq J$ be two closed ideals in $L^1(G)$ such that $h(I) = h(J) = \Omega$. Then*

(1) *$\exists K$ (closed ideal) $\subset L^1(G)$ such that $I \subsetneqq K \subsetneqq J$.*
(2) *If each $w \in \Omega$ has a countable open basis, then there is a descending chain of closed ideals K_n of $L^1(G)$ such that $K_n \subset J\,(n = 1, 2, \cdots)$ and $\bigcap_{n=1} K_n = I$.*

Proof. (1) See [8]. (2) Take $f \in J \setminus I$, and set $E = \{\gamma \in \Gamma : f \notin \tilde{I}_{(\gamma)}\}$. Then $E \neq \phi$. To see this, note that we can find a sequence $\{v_n\}$ in $L^1(G)$ such that the support of $\hat{v}_n$ is compact and $\|f * v_n - f\| \to 0$ as $n \to \infty$. So if $E = \phi$, then $f * v_n \in \tilde{I}_{(\gamma)}$ for every $\gamma \in \Gamma$, hence $f * v_n \in I$. But since I is closed, $f \in I$, which is a contradiction. Thus $E \neq \phi$, and it follows that E is a perfect subset of Ω.

Now choose a $\gamma \in E$, and let $U_\circ$ be a compact neighborhood of γ. Since each point in Ω has a countable open basis and E is a perfect subset of Ω, there is a sequence of compact neighborhood U_n of γ satisfying the following conditions :

(i) $U_n \subset U_{n-1}^\circ\,(n = 1, 2, \cdots)$ and U_{n-1} contains an open set V_n such that $V_n \cap U_n = \phi$ and $V_n \cap E \neq \phi$
(ii) $(\bigcap_{n=1} U_n) \bigcap E = \{\gamma\}$.

For $n = 1, 2, \cdots$, take a function $k_n \in L^1(G)$ such that $\hat{k}_n = 1$ on U_{2n} and $\hat{k}_n = 0$ on U_{2n-1}^c, and define K_n to be the closed ideal in $L^1(G)$ generated by $f * k_n$ and the functions in I.

Repeating this procedure for fixed n, we see that $K_n \subset J$, $K_{n+1} \subsetneqq K_n$.

For $\bigcap_{n=1} K_n = I$, we note from (2) that $f \in \tilde{I}_{(\eta)}$ for every $f \in \bigcap_{n=1} K_n$ and $\eta \in E \setminus \{\gamma\}$, and this completes the proof.

We now improve the (1) of Proposition 3.2.

LEMMA 3.3. *Let I and J be the two ideals of a commutative Banach algebra A, and define $IJ = \{fg: f \in I, g \in J\}$.*

(1) *If $I \subset J$ and no ideals of A lie strictly between them, then either $AJ \subset I$ or $MJ \subset I$ for a maximal ideal M containing I.*
(2) *If I and J are closed and either has approximate identity, then $IJ = I \bigcap J$.*

Proof. (1) Let $M = \{f \in A: fJ \subset I\}$. Then M is an ideal of A.

(i) If $A = M$, then the result is clear.

(ii) Suppose $A \neq M$, and take $g \in A \setminus M$. Then $kg \notin I$ for some $k \in J$, and $I + kgA + \mathbb{C}kg$ is an ideal such that $I \subsetneq I + kgA + \mathbb{C}kg \subset J$. So we must have $J = I + kgA + \mathbb{C}kg$, and for each $f \in A$, $k(f - ga - \alpha g) \in I$ for some $a \in A$, $\alpha \in \mathbb{C}$. Since $k \notin I$, $J = I + kA + \mathbb{C}k$, so that

$$(f - ga - \alpha g)J = (f - ga - \alpha g)I + k(f - ga - \alpha g)A + \mathbb{C}k(f - ga - \alpha g) \subset I.$$

Hence $f - ga - \alpha g \in M$, that is, $A = M + gA + Fg$. Thus M is a maximal ideal of A.

(2) The result easily follows from the Cohen's factrization theorem (Lemma 1.1).

Let A be a commutative Banach algebra with character space $\Delta(A)$. A is said to be <u>regular</u> if for a closed set $E \subset \Delta(A)$ and $\tau \in \Delta(A) \setminus E$ there is some $f \in A$ such that $\hat{f}(\tau) = 1$ and $\hat{f}(w) = 0$, $w \in E$.

LEMMA 3.4. *If G is a locally compact abelian group, then $L^1(G)$ is regular.*

Proof. See [9] or [11].

THEOREM 3.5. *Let G be a locally compact abelian group. If $I \subsetneq J$ are ideals of $L^1(G)$ such that $h(I) = h(J)$ and one of them is closed, then there is an ideal of $L^1(G)$ strictly between them.*

Proof. We first note that $L^1(G)$ has approximate identity and so does each maximal ideal of $L^1(G)$ [11, P.151]. For each $\gamma \in h(I)$, let $I_\gamma = \{f \in L^1(G): \hat{f}(\gamma) = 0\}$. Then I_γ is a closed maximal ideal of $L^1(G)$ containing I, and since $h(I) = h(J)$ it also contains J. Further, every maximal ideal of $L^1(G)$ which contains I and J are of the form I_γ for a $\gamma \in h(I)$.

Now suppose there are no ideals strictly between I and J. Then, by Lemma 3.3 (1), we have $L^1(G) * J \subset I$ or $M * J \subset I$ for some maximal ideal M containing I and hence $J \subset M$. If J is closed, then since M and $L^1(G)$ have approximate identity, by Lemma 3.3 (2),

$$J = J \cap L^1(G) = L^1(G) * J \subset I$$

or

$$J = J \cap M = M * J \subset I.$$

Thus in either case we have $J = I$, which is a contradiction.

If I is closed, then again because $L^1(G)$ and M have approximate identities, we have

$$J \subset \overline{L^1(G) * J} \subset \bar{I} = I \text{ or } J \subset \overline{M * J} \subset \bar{I} = I,$$

and hence $J = I$, which is also a contradiction.

In order to develop a chain of ideals in $L^1(G)$, we note that, for a closed set Ω in Γ, $K(\Omega)$, $F(\Omega)$ and $H(\Omega)$ (defined in §2) are ideals of $L^1(G)$ and $K(\Omega) \subset F(\Omega) \subset H(\Omega)$.

COROLLARY 3.6. *Let G be a locally compact abelian group and Ω be a closed set in Γ. If I is an ideal of $L^1(G)$ with $K(\Omega) \subsetneqq I \subsetneqq H(\Omega)$, there exist ideals I_* and I^* such that $K(\Omega) \subset I_* \subset I \subset I^* \subset H(\Omega)$ with all inclusions proper.*

Proof. Since $L^1(G)$ is regular (Lemma 3.4), $h[K(\Omega)] = \Omega = h[H(\Omega)]$ and every ideal of $L^1(G)$ whose hull is Ω lies between $K(\Omega)$ and $H(\Omega)$ [11, P.84]. Further, since $H(\Omega)$ is closed and $h(I) = h[H(\Omega)]$, it follows from the Theorem 3.5 that there exists an ideal I^* such that $I \subsetneqq I^* \subsetneqq H(\Omega)$. For the existence of I_*, suppose that there is no such an ideal. Then since every maximal ideal of $L^1(G)$ containing $K(\Omega)$ also contains I, by Lemma 3.3 (1), $I^2 \subset K(\Omega)$. But $K(\Omega)$ is semiprime, that is, $f^2 \in I^2 \subset K(\Omega) \Rightarrow f \in K(\Omega)$, thus we have $I = K(\Omega)$, which is a contradiction.

REMARK. *By repeated use of the above result we see that through any such I we may have infinite ascending and descending chains of ideals of $L^1(G)$ whose hull is Ω.*

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