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SOME HOMOMORPHISMS OF BCI-ALGEBRAS

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In this paper, some homomorphisms of BCI-algebras are discussed. Recall that a BCI-algebra is a nonempty set X with a binary operation $*$ and a constant 0 satisfying the axioms:

BCI-1 $(x * y) * (x * z) \leq z * y$,

BCI-2 $x * (x * y) \leq y$,

BCI-3 $x \leq x$,

BCI-4 $x \leq y$ and $y \leq x$ imply $x = y$,

BCI-5 $x \leq y$ if and only if $x = y$,

for all $x, y, z \in X$.

The following identities hold for any BCI-algebra X :

(1) $x * 0 = x$,

(2) $(x * y) * z = (x * z) * y$,

(3) $0 * (x * y) = (0 * x) * (0 * y)$,

(4) $0 * (0 * (0 * x)) = 0 * x$.

A nonempty subset I of a BCI-algebra X is called an ideal of X if (i) $0 \in I$, (ii) $x * y \in I$ and $y \in I$ imply $x \in I$. For any BCI-algebra X, the set $X_+ = \{x \in X | 0 \leq X\}$ is called the BCK-part of X. A mapping $f : X \to Y$ of BCI-algebras is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

For any BCI-algebra X, the set $X_p = \{x \in X | 0 * (0 * x) = x\}$ is called the p-semisimple part of X, and the set $G(X) = \{x \in X | 0 * x = x\}$ is said to be the BCI-G part of X.

The element $0 \in X$ is said to be a weak unit of X if $0 * x \leq x$ for all $x \in X$.

LEMMA 1 ([3]). *Let X be a BCI-algebra. Then the following conditions are equivalent:*

(5) *X is p-semisimple.*

(6) $0 * (0 * x) = x$.

(7) $x * z = y * z$ *implies* $x = y$.

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(8) $(x * y) * (w * z) = (x * w) * (y * z)$.
(9) $x * y = 0$ *implies* $x = y$.

Using (3), we have

LEMMA 2. *For any homomorphism $f : X \to Y$ of BCI-algebras, define a map $\alpha : X \to Y$ by $\alpha(x) = 0 * f(x)$ for all $x \in X$. Then α is a homomorphism.*

In what follows, X and Y always denote BCI-algebras, f denote any homomorphism from X to Y, and $\alpha : X \to Y$ denote a homomorphism defined as in Lemma 2.

PROPOSITION 1. *(i) If f is one to one, $\ker(\alpha)$ is equal to the BCK-part X_+ of X. (ii) If f is onto, then $Im(\alpha)$ is equal to the p-semisimple part Y_p of Y.*

Proof. (i) Assume that f is one to one. Then

$$\begin{aligned}\ker(\alpha) &= \{x \in X | \alpha(x) = 0\} \\ &= \{x \in X | 0 * f(x) = 0\} \\ &= \{x \in X | f(0 * x) = f(0)\} \\ &= \{x \in X | 0 * x = 0\} \\ &= X_+.\end{aligned}$$

(ii) Assume that f is onto. Let $y \in Im(\alpha)$. Then $y = \alpha(x)$ for some $x \in X$ and

$$\begin{aligned}0 * (0 * y) &= 0 * (0 * \alpha(x)) \\ &= 0 * (0 * (0 * f(x))) \\ &= 0 * f(x) \quad [\text{by } (4)] \\ &= \alpha(x) \\ &= y.\end{aligned}$$

Hence $y \in Y_p$.

Conversely, suppose that $y \in Y_p$. Since $0 * y \in Y$ and f is onto, there exist $x \in X$ such that $f(x) = 0 * y$. Then

$$\begin{aligned}y &= 0 * (0 * y) \\ &= 0 * f(x) \\ &= \alpha(x) \in Im(\alpha).\end{aligned}$$

This completes the proof.

THEOREM 1. *Let* $A = \{x \in X | \alpha(x) = f(x)\}$. *Then* A *is a subalgebra of* X.

Proof. Let $a, b \in A$. Then $\alpha(a) = f(a)$ and $\alpha(b) = f(b)$. Thus we have

$$\begin{aligned} \alpha(a * b) &= 0 * f(a * b) \\ &= 0 * (f(a) * f(b)) \\ &= (0 * f(a)) * (0 * f(b)) \quad \text{[by (3)]} \\ &= \alpha(a) * \alpha(b) \\ &= f(a) * f(b) \\ &= f(a * b). \end{aligned}$$

Therefore $a * b \in A$, which completes the proof.

COROLLARY 1. *If* f *is one to one, then* $A = G(X)$.

THEOREM 2. *Let* $B = \{f(x) \in Y | \alpha(x) = f(x)$ *for some* $x \in X\}$. *Then*
(a) $B \subset Y_p$.
(b) B *is a subalgebra of* Y_p *(and also* Y*).*
(c) B *is an ideal of* Y_p.

Proof. (a) is obvious.

(b) Let $x, x' \in X$ be such that $f(x), f(x') \in B$. Then $\alpha(x) = f(x)$ and $\alpha(x') = f(x')$. Now

$$\begin{aligned} \alpha(x * x') &= \alpha(x) * \alpha(x') \\ &= f(x) * f(x') \\ &= f(x * x'). \end{aligned}$$

Thus $f(x) * f(x') = f(x * x') \in B$.

(c) Since $\alpha(0) = f(0)$, $f(0) = 0 \in B$. Let $x, y \in X$ be such that $f(x) * f(y), f(y) \in B$. Then

$$\begin{aligned} f(x) * f(y) &= f(x * y) \\ &= \alpha(x * y) \\ &= \alpha(x) * \alpha(y) \\ &= \alpha(x) * f(y). \end{aligned}$$

From Lemma 1(7), it follows that $f(x) = \alpha(x)$, so that $f(x) \in B$.

COROLLARY 2. *If Y is a p-semisimple, then B is an ideal of Y.*

REMARK 1. The reverse inclusion of (a) in Theorem 2 is not true in general. For instance, let $X = \{0, a, x, y, z\}$ and $*$ is defined by:

$*$	0	a	x	y	z
0	0	0	z	z	x
a	a	0	z	z	x
x	x	x	0	0	z
y	y	x	a	0	z
z	z	z	x	x	0

Then X is a BCI-algebra([3]). Consider the identity map $1_X : X \to X$. Then by routine calculation, $B = \{0\}$ and $X_p = \{0, x, z\}$.

THEOREM 3. *If Y has a weak unit 0 and if f is onto, then $Y_p = B$.*

Proof. It is sufficient to show that $Y_p \subset B$. If $y \in Y_p$, then $y = f(x)$ for some $x \in X$. Since 0 is a weak unit of Y, therefore

$$0 = (0 * y) * y = (0 * f(x)) * f(x) = \alpha(x) * f(x).$$

As $f(x) \in Y_p$, it follows from (4) that $\alpha(x) \in Y_p$. Hence by Lemma 1(9), we have that $\alpha(x) = f(x)$. This means that $f(x) = y \in B$, so that $Y_p \subset B$. The proof is complete.

LEMMA 4. *Assume that Y is p-semisimple. Then the maps $q_1, q_2 : X \to Y$ defined by $q_1(x) = f(x) * (0 * f(x))$ and $q_2(x) = (0 * f(x)) * f(x)$ for every $x \in X$ are homomorphisms.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned} q_1(x * y) &= f(x * y) * (0 * f(x * y)) \\ &= (f(x) * f(y)) * (0 * (f(x) * f(y))) \\ &= (f(x) * f(y)) * ((0 * f(x)) * (0 * f(y))) \quad \text{[by (3)]} \\ &= (f(x) * (0 * f(x))) * (f(y) * (0 * f(y))) \quad \text{[by (8)]} \\ &= q_1(x) * q_1(y) \end{aligned}$$

and

$$\begin{aligned} q_2(x * y) &= (0 * f(x * y)) * f(x * y) \\ &= (0 * (f(x) * f(y))) * (f(x) * f(y)) \\ &= ((0 * f(x)) * (0 * f(y))) * (f(x) * f(y)) \quad \text{[by (3)]} \\ &= ((0 * f(x)) * f(x)) * ((0 * f(y)) * f(y)) \quad \text{[by (8)]} \\ &= q_2(x) * q_2(y). \end{aligned}$$

Hence q_1 and q_2 are homomorphisms.

THEOREM 4. *Assume that Y is p-semisimple. Then the following graphs*

$$\begin{array}{ccc} X & \xrightarrow{h_1} & X \\ \downarrow{\scriptstyle q_1} & \searrow^{q_2} & \downarrow{\scriptstyle q_1} \\ Y & \xrightarrow{h_2} & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{h_1} & X \\ \downarrow{\scriptstyle q_2} & \searrow^{q_1} & \downarrow{\scriptstyle q_2} \\ Y & \xrightarrow{h_2} & Y \end{array}$$

are commutative, where $h_1 : X \to X, x \mapsto 0 * x$ and $h_2 : Y \to Y, y \mapsto 0 * y$ are endomorphisms of X and Y, respectively.

Proof. For any $x \in X$, we have

$$\begin{aligned} (h_2 \circ q_1)(x) &= h_2(f(x) * (0 * f(x))) \\ &= 0 * (f(x) * (0 * f(x))) \\ &= (0 * f(x)) * (0 * (0 * f(x))) \quad \text{[by (3)]} \\ &= (0 * f(x)) * f(x) \quad \text{[by (6)]} \\ &= q_2(x). \end{aligned}$$

Similarly, we have $q_1 \circ h_1 = q_2$ and $h_2 \circ q_2 = q_1 = q_2 \circ h_1$.

PROPOSITION 2. *If Y is p-semisimple, then $q_1(x) * f(x) = f(x)$ for all $x \in X$.*

Proof. For any $x \in X$, we have

$$\begin{aligned} q_1(x) * f(x) &= (f(x) * (0 * f(x))) * f(x) \\ &= (f(x) * f(x)) * (0 * f(x)) \\ &= 0 * (0 * f(x)) \\ &= f(x). \end{aligned}$$

THEOREM 5. *Assume that Y is p-semisimple. Then (i) $Im(q_1)$ is of the form*

$$Im(q_1) = \{y \in Y | y * f(x) = f(x) \text{ for some } x \in X\} = D(say).$$

(ii) If $Im(q_1) = Im(q_2)$, then D is an ideal of Y.

Proof. (i) If $y \in Im(q_1)$, then $y = q_1(x)$ for some $x \in X$. It follows from Proposition 2 that

$$y * f(x) = q_1(x) * f(x) = f(x) \quad \text{for some } x \in X.$$

Hence $y \in D$, and so $Im(q_1) \subset D$. Conversely, if $y \in D$, then $y * f(x) = f(x)$ for some $x \in X$. It follows from (1) and (8) that

$$\begin{aligned} y &= (y * 0) * ((y * f(x)) * f(x)) \\ &= (y * (y * f(x))) * (0 * f(x)) \\ &= (y * f(x)) * (0 * f(x)) \\ &= f(x) * (0 * f(x)) \\ &= q_1(x) \in Im(q_1). \end{aligned}$$

Hence $D \subset Im(q_1)$.

(ii) Clearly $0 \in D$. Let $a, b \in Y$ be such that $a * b, b \in D$. Then there exist $x, y \in X$ such that $q_1(x) = a * b, q_1(y) = b$. Thus we have

$$\begin{aligned} a &= (a * 0) * (b * b) \quad \text{[by BCI-3 and (1)]} \\ &= (a * b) * (0 * b) \quad \text{[by (8)]} \\ &= q_1(x) * h_2(q_1(y)) \\ &= q_1(x) * q_2(y) \quad \text{[by Theorem 4]} \\ &= q_1(x) * q_1(z) \quad \text{for some } z \in X \\ &= q_1(x * z) \in D. \end{aligned}$$

Therefore D is an ideal of Y.

THEOREM 6. *Assume that Y is p-semisimple. Then $D \subset B$ if and only if $q_1 = q_2$.*

Proof. Suppose that $D \subset B$. Then $q_1(x) \in D \subset B$ for all $x \in X$. It follows that

$$q_2(x) = (h_2 \circ q_1)(x) = h_2(q_1(x)) = q_1(x).$$

Hence $q_1 = q_2$.

Conversely, assume that $q_1 = q_2$. If $a \in D$, then $a = q_1(x)$ for some $x \in X$, and

$$\begin{aligned} 0 * a &= 0 * q_1(x) \\ &= 0 * (f(x) * (0 * f(x))) \\ &= (0 * f(x)) * (0 * (0 * f(x))) \\ &= (0 * f(x)) * f(x) \\ &= q_2(x) \\ &= q_1(x) \\ &= a. \end{aligned}$$

Therefore $a \in B$. This completes the proof.

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