Honam Mathematical J. 16(1994), No. 1,pp. 67–70

FUZZY CHARACTERISTIC Γ–IDEALS OF A Γ–RING

SUNG-MIN HONG AND YOUNG-BAE JUN

Dept. of Mathematics, Gyeongsang National University, Chinju 660-701, Korea.

1. Introduction

The notion of a Γ-ring was first introduced by Nobusawa [6]. Since then, many researchers have investigated various properties of this Γ-ring. Any ring can be regarded as a Γ-ring by suitably choosing Γ. Many fundamental results in ring theory have been extended to Γ-rings. The notion of fuzzy sets was introduced by Zadeh [8]. The fuzzy set theory developed by Zadeh himself and others has found many applications in the domain of mathematics and elsewhere. Mukherjee and Sen [5] introduced and studied the notion of fuzzy ideals in a ring. In [3], the first author together with Lee applied the concept of fuzzy sets to the elementary theory of Γ-rings. Also the second author [2] introduced the concept of a fuzzy prime ideal in a Γ-ring, and investigated its properties.

In this paper we introduce the concept of fuzzy characteristic Γ-ideals of a Γ-ring, and we show that a fuzzy characteristic Γ-ideal is characterized in terms of its level Γ-ideals.

2. Preliminaries

Let us recall some definitions and results, which are necessary for development of the paper.

Let $M = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be additive abelian groups. If for all x, y, z in M and all α, β in Γ, the following conditions are satisfied

(1) $x\alpha y$ is an element of M,

(2) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$,

(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

then, following Barnes [1], M is called a Γ-ring.

Through this paper M denotes a Γ-ring, and 0_M denotes the zero element of M.

Received September 7, 1993.

A subset A of M is a left (right) ideal of M ([1]) if A is an additive subgroup of M and

$$M\Gamma A = \{x\alpha y | x \in M, \alpha \in \Gamma, y \in A\}(A\Gamma M)$$

is contained in A. If A is both a left and a right ideal, then it is called a Γ-ideal of M.

A mapping $\theta : M \to M'$ of Γ-rings is called a Γ-homomorphism ([1]) if $\theta(x+y) = \theta(x) + \theta(y)$ and $\theta(x\alpha y) = \theta(x)\alpha\theta(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Throughout $Aut(M)$ will denote the set of all automorphisms of M. We now review some fuzzy logic concepts. We refer the reader to [2] and [3] for complete details. A fuzzy set in a set S is a function $\mu : S \to [0,1]$. Let μ be a fuzzy set in a set S. For $t \in [0,1]$, the set $\mu_t = \{x \in S | \mu(x) \geq t\}$ is called a level subset of μ.

A fuzzy set μ in M is called a fuzzy left (right) ideal of M ([3]) if

(4) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,

(5) $\mu(x\alpha y) \geq \mu(y) \quad (\mu(x\alpha y) \geq \mu(x))$,

for all $x, y \in M$ and all $\alpha \in \Gamma$. If μ is both a fuzzy left and a fuzzy right ideal of M, then it is called a fuzzy Γ-ideal of M.

We note that μ is a fuzzy Γ-ideal of M if and only if

(4) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,

(6) $\mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\}$,

for all $x, y \in M$ and $\alpha \in \Gamma$.

Let μ be a fuzzy set in M. It was shown in [3] that

(a) if μ is a fuzzy Γ-ideal of M, then μ_t is a Γ-ideal of M for all $t \in [0, \mu(0_M)]$, which is called the level Γ-ideal of M.

(b) if μ_t is a Γ-ideal of M for all $t \in Im(\mu)$, then μ is a fuzzy Γ-ideal of M, where $Im(\mu)$ is the image of μ.

3. Fuzzy characteristic Γ-ideals

DEFINITION 3.1. If μ is a fuzzy Γ-ideal of M and θ is a mapping from M into itself, we define a mapping $\mu^\theta : M \to [0,1]$ by $\mu^\theta(x) = \mu(\theta(x))$ for all $x \in M$.

THEOREM 3.2. *If μ is a fuzzy Γ-ideal of M and θ is a Γ-endomorphism of M, then μ^θ is a fuzzy Γ-ideal of M.*

proof. Assume that μ is a fuzzy Γ-ideal of M. Let $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}\mu^{\theta}(x-y) &= \mu(\theta(x-y)) = \mu(\theta(x)-\theta(y)) \\ &\geq \min\{\mu(\theta(x)), \mu(\theta(y))\} \\ &= \min\{\mu^{\theta}(x), \mu^{\theta}(y)\}\end{aligned}$$

and

$$\begin{aligned}\mu^{\theta}(x\alpha y) &= \mu(\theta(x\alpha y)) = \mu(\theta(x)\alpha\theta(y)) \\ &\geq \max\{\mu(\theta(x)), \mu(\theta(y))\} \\ &= \max\{\mu^{\theta}(x), \mu^{\theta}(y)\}.\end{aligned}$$

Hence μ^{θ} is a fuzzy Γ-ideal of M.

Combining Theorem 3.2 and [3, Proposition 1], we have the following corollary.

COROLLARY 3.3. *If μ is a fuzzy Γ-ideal of M and θ is a Γ-endomorphism of M, then*
(a) $\mu^{\theta}(0_M) \geq \mu^{\theta}(x)$,
(b) $\mu^{\theta}(-x) = \mu^{\theta}(x)$,
(c) $\mu^{\theta}(x-y) = \mu^{\theta}(0_M)$ *implies* $\mu^{\theta}(x) = \mu^{\theta}(y)$,
for all $x, y \in M$.

DEFINITION 3.4. A Γ-ideal K of M is said to be characteristic if $\theta(K) = K$ for all $\theta \in Aut(M)$.

DEFINITION 3.5. A fuzzy Γ-ideal μ of M is said to be fuzzy characteristic if $\mu(\theta(x)) = \mu(x)$ for all $x \in M$ and $\theta \in Aut(M)$.

THEOREM 3.6. *Let μ be a fuzzy characteristic Γ-ideal of M. Then each level Γ-ideal of μ is a characteristic Γ-ideal of M.*

Proof. Assume that μ is a fuzzy characteristic Γ-ideal of M. It is sufficient to show that $\theta(\mu_t) = \mu_t$, where $t \in Im(\mu)$. Let $t \in Im(\mu)$, $\theta \in Aut(M)$ and $x \in \mu_t$. Since μ is fuzzy characteristic, we have $\mu(\theta(x)) = \mu(x) \geq t$. It follows that $\theta(x) \in \mu_t$ and hence $\theta(\mu_t) \subseteq \mu_t$. To prove the reverse inclusion, let $y \in \mu_t$ and let $x \in M$ be such that $\theta(x) = y$. Then $\mu(x) = \mu(\theta(x)) = \mu(y) \geq t$, whence $x \in \mu_t$. This implies that $y = \theta(x) \in \theta(\mu_t)$, so that $\mu_t \subseteq \theta(\mu_t)$. Thus $\mu_t, t \in Im(\mu)$, is a characteristic Γ-ideal of M.

The following lemma is obvious, and we omit the proof.

LEMMA 3.7. *Let μ be a fuzzy Γ-ideal of M and let $x \in M$. Then $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$.*

Now we prove the converse of Theorem 3.6.

THEOREM 3.8. *Let μ be a fuzzy Γ-ideal of M. If each level Γ-ideal of μ is characteristic, then μ is a fuzzy characteristic Γ-ideal of M.*

Proof. Let μ be a fuzzy Γ-ideal of M. Let $x \in M$, $\theta \in Aut(M)$ and $\mu(x) = t$. By Lemma 3.7, $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$. From hypothesis, it follows that $\theta(\mu_t) = \mu_t$. Thus $\theta(x) \in \theta(\mu_t) = \mu_t$, and so $\mu(\theta(x)) \geq t$. Let $\mu(\theta(x)) = s$ and assume that $s > t$. Then $\theta(x) \in \mu_s = \theta(\mu_s)$. Since θ is one-to-one, it follows that $x \in \mu_s$. This is a contradiction. Hence $\mu(\theta(x)) = t = \mu(x)$, showing that μ is a fuzzy characteristic Γ-ideal of M.

References

1. W. E. Barnes, *On the Γ-rings of Nobusawa*, Pacific J. Math. **18** (1966), 411-422.
2. Y. B. Jun, *On fuzzy prime ideals of Γ-rings*, Soochow J. Math. (to appear).
3. Y. B. Jun and C. Y. Lee, *Fuzzy Γ-rings*, Pusan Kyongnam Math. J. **8** (1992), 163-170.
4. N. Kuroki, *On fuzzy ideals and fuzzy bi-ideals in semigroups*, Fuzzy Sets and Systems **5** (1981), 203-215.
5. T. K. Mukherjee and M. K. Sen, *On fuzzy ideals of a ring I*, Fuzzy Sets and Systems **21** (1987), 99-104.
6. N. Nobusawa, *On a generalization of the ring theory*, Osaka J. Math. **1** (1964), 81-89.
7. T. S. Ravisankar and U. S. Shukla, *Structure of Γ-rings*, Pacific J. Math. **80** (1979), 537-559.
8. L. A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338-353.