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ON THE VERBALLY PRIME T–IDEALS OF THE FREE ASSOCIATIVE ALGEBRA

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1. Introduction

The purpose of this article is to present some results on the structure of a p.i. Algebra. Especially the author will discuss the verbally prime algebras related to the inclusions of their T-ideals. These results follow fairly directly from Kemer's works ([6]) and the author was communicated by A. Berele ([4]).

We shall fix some notations.

Let $K < X >$ be the free associative algebra over a field K of characteristic zero, generated by the countable set $X = \{x_1, x_2, \dots\}$, and let A be an arbitrary associative algebra over K. An ideal I in $K < X >$ is called a T-ideal if $\varphi(I) \subseteq I$ for all endomorphisms φ of $K < X >$.

For example, $\mathrm{T}(A) = \{\ f(x) \in K < X > |f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in A\ \}$ is a T-ideal of $K < X >$.

S.A. Amitsur showed that an algebra A is p.i. equivalent to the algebra $M_n(K)$ of $n \times n$ matrices over K if and only if the set $\mathrm{T}(A)$ of all identities for A is a prime T-ideal in $K < X >$. Thus $f(X_1, \dots, X_n)\ g(X_1, \dots, X_k)$ in $K < X >$ is an identity for A if and only if $f(X_1, \dots, X_n)$ or g $(X_1, \dots, X_k)$ is an identity for A. ([1]). In study of T-ideals, A.R. Kemer generalized this theorem on the variety which is the class of all algebras satisfying a given system of polynomial identities $\{f_i(X_1, \dots, X_n) = 0 | i \in I\}$ ([6]).

Given an algebra A over K we will write $Id(A)$ to denote the T-ideal of $K < X >$ of identities for A; and given a T-ideal I in $K < X >$ we will write $I(A)$ for the ideal of evaluations of polynomials in I on A. The ideal $I(A)$ will be called a T-ideal of A. Finally, we will say the two K-algebras A and B are p.i. equivalent if $Id(A) = Id(B)$ and denote by $A \sim B$.

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2. Verbally semiprime p.i. algebras and verbally prime p.i. algebras

DEFINITION. A p.i. algebra A is called verbally prime if whenever for any two T-ideals I, J of identities of A the equation $IJ = 0$ implies $I = 0$ or $J = 0$.

Note : A is a verbally prime p.i. algebra if whenever $f\ (X_1, \ldots, X_n)$ $g\ (\ X_{n+1},\ \ldots,\ X_m\ )$ is an identity for A then either $f(X_1, \ldots, X_n)$ or $g\ (\ X_{n+1},\ \ldots,\ X_m\ )$ is a identity for A, where $f\ (X_1, \ldots, X_n)$ and g $(X_{n+1}, \ldots, X_m)$ are evaluated at disjoint of indeterminates.

DEFINITION. A p.i. algebra A is said to be verbally semiprime if for any T-ideal I of A, $I^n = 0$ for some positive integer n implies $I = 0$.

Note : A is a verbally semiprime p.i. algebra if whenever $f^n(X_1, \ldots, X_n)$ is an identity of A then $f(X_1, \ldots, X_n)$, is an identity of A, for some n, then $f(X_1, \ldots, X_n)$ is an identity of A.

We consider the natural $\mathbb{Z}_2$-grading for the Grassmann algebra $E = E_0 + E_1$, where E_0 $[E_1]$ is the subspace of E spanned by all monomials involving an even [odd] number of elements e_i's from the vector space.

Let p, q be positive integers, $p \geq q$. Let $M_{p+q}(E) = M_{p+q}(K) \otimes E$ be the algebra of $(p+q) \times (p+q)$ matrices with entries from E. One denotes by $M_{p,q}$ the subalgebra of $M_{p+q}(E)$ generated by all matrices of the form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{11} is a $p \times p$ matrix with entries in E_0, A_{22} is a $q \times q$ matrix with entries E_0, and A_{12} and A_{21} have entries in E_1.

The description of all verbally prime and verbally semiprime p.i. albebras is given by the next two results ([6]).

THEOREM 2.1. *Every verbally semiprime p.i. algebra must be p.i. equivalent to a finite direct sum of verbally prime p.i. algebras.*

THEOREM 2.2. *Every verbally prime p.i. algebra must be p.i. equivalent to one the following algebras*

(a) $M_n(K)$, *the* $n \times n$ *matrices over a field* K.

(b) $M_n(E)$, *the* $n \times n$ *matrices over the infinite dimensional Grassman algebra* E

(c) $M_{p,q}$, $p \geq q \geq 1$ *,the subalgebra of* $M_{p+q}(E)$.

3. Main Theorem

LEMMA 3.1. *Let I be a T-ideal in $K < X >$. Then the following holds.*
(1) If $A \sim B$ are p.i. equivalent algebras over K then $I(A) \sim I(B)$.
(2) If A is a direct sum $A = \sum_i A_i$, then $I(A) = \sum_i I(A_i)$.
(3) If A is verbally prime, then either $I(A) = 0$ or $I(A) \sim A$.

Proof. (1) A polynomial $f(x_1, x_2, \ldots, x_n)$ is an identity for a T-ideal $I(A)$ of A if and only if $f(g_1(x), \ldots, g_n(x))$ is an identity for A when $g_1(x), \ldots, g_n(x) \in I$. Since $Id(A) = Id(B)$, it follows $Id(I(A)) = Id(I(B))$ for $I(A) \triangleleft A$ and $I(B) \triangleleft B$. Hence $I(A) \sim I(B)$.
(2) Suppose $A = A_1 \oplus \cdots \oplus A_q$. Then

$$I(A) = I(A_1 \oplus \cdots \oplus A_q) = I(A_1) \oplus \cdots \oplus (A_q) = \sum_i I(A_i).$$

(3) We may, without loss of generality, assume that $A = M_n(K)$. Hence A is simple and the only ideals of A are A itself and 0. If $A = 0$ then $I(A) = I(0) = 0$ so $I(A) = 0$
If $A = M_n(K)$ then $I(A) = I(M_n(K)) \triangleleft M_n(K)$ and so $I(M_n(K)) = I(A) \sim A = M_n(K)$ since $M_n(K)$ is simple.

THEOREM 3.2. *Suppose that the following conditions holds for two algebras $A = A_1 \oplus \cdots \oplus A_n$ and $B = B_1 \oplus \cdots \oplus B_m$.*
(1) $A_1 \ldots A_n, B_1, \ldots, B_M$ are verbally prime,
(2) $A = A_1 \oplus \cdots \oplus A_n \sim B = B_1 \oplus \cdots \oplus B_m$, and
(3) $A_1 \oplus \cdots \oplus \widehat{A}_i \oplus \cdots \oplus A_n$ is not p.i. equivalent to A, and $B_1 \oplus \cdots \oplus \widehat{B}_j \oplus \cdots \oplus B_m$ is not p.i. equivalent to B.
Then the following hold
(a) $n = m$ and
(b) there is a permutation $\sigma \in S_m$ such that $A_i \sim A_{\sigma(i)}$ for all $i = 1, 2, \ldots, n$.

Proof. (a) Since $Id(A_1 \oplus \cdots \oplus A_n) = Id(b_1 \oplus \cdots \oplus B_m)$ and $Id(A_1 \oplus \cdots \oplus \widehat{A}_i \oplus \cdots A_n) \neq Id(A_1 \oplus \cdots \oplus A_n)$, By hypothesis, it follows that $Id(B_1 \oplus \cdots \oplus \widehat{B}_j \oplus \cdots \oplus B_m) \neq Id(B_1 \oplus \cdots \oplus B_m)$ $n = m$.
(b) Let $Id(A_i) = I_i$ be a T-ideal of $K < X >$ for each i and consider $I_i(A) \sim I_i(B)$. *i.e.*, $Id(I_i(A)) = Id(I_i(B))$. Since $I_i(A_i) = 0, A = A_1 \oplus \cdots \oplus A_n$ is not p.i. equivalent to

$$\begin{aligned} I_i(A) &= I_i(A_1 \oplus \cdots \oplus A_i \oplus \cdots \oplus A_n) \\ &= I_i(A_i) \oplus I_i(A_2) \oplus \cdots \oplus I_i(A_i) \oplus \cdots \oplus I_i(A_n), \end{aligned}$$

by hypothesis, $A \sim B$. Hence there are $I_i(B_j) = 0$ for some j. So for $i = 1, 2, \ldots, n$ there is a $\sigma(i)$ such that $Id(A_i) \subset Id(B_{\sigma(i)})$. Similerly there is a function τ such that $Id(A_{\tau(j)}) \subset Id(B_j)$ for $n = 1, 2, \ldots m$.
Since $Id(A_i) \subset Id(B_{\sigma(i)})$ and $Id(A_{\tau(j)}) \subset Id(B_j)$
we have $Id(A_{\tau(\sigma(i))}) \subset Id(B_{\sigma(i)})$ $Id(A_i) \subset Id(A_{\tau(\sigma(i))})$ or $Id(A_{\tau(\sigma(i))}) \subset Id(A_i)$
for all $i = 1, 2, \ldots n$. But by hypothesis $Id(A_k)$ is not contained in $Id(A_i)$ for any $k \neq i$, and so $(\tau\sigma)(i) = i$ for all $i = 1, 2, \ldots, n$. Similarly, $(\sigma\tau)(j) = j$ for all j so $n = m$. Hence there exists a permutation $\sigma \in S_n$ such that $A_i \sim B_{\sigma(i)}$ for all $i = 1, 2, \ldots n$. $\{1, 2, \ldots, n\}$.

THEOREM 3.3. *Assume that $A_1, \ldots, A_n$ are verbally prime and that $A_1 \oplus \cdots \oplus A_n \sim A_1 \oplus \cdots \oplus A_{n-1}$. Then $Id(A_i) \subset Id(A_n)$ for some $i < n$.*

Proof. Consider $A_2 \oplus \cdots \oplus A_{n-1}$. If $A_2 \oplus \cdots \oplus A_n \sim A_2 \oplus \cdots \oplus A_{n-1}$, then we may delete A_1 from the sum and work with a smaller n. Hence we may assume that $A_1 \oplus \cdots \oplus A_{n-1}$ is of minimal length subject to the hypothesis of Lemma. Let $I = Id(A_1)$ and calculate

$$I(A_1 \oplus \cdots \oplus A_n) \sim I(A_1 \oplus \cdots \oplus A_{n-1})$$

so $I(A_2) \oplus \cdots \oplus I(A_{n-1}) \oplus I(A_n) \sim I(A_2) \oplus \cdots \oplus I(A_{n-1})$ for each $i = 2, \ldots, n$, and so $I(A_i)$ is equivalent to either A_i or 0. Hence, by the minimality we have $I(A_n) = 0$ and $Id(A_1) \subset id(A_n)$ as claimed.

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