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ON CERTAIN CLASSES OF CONVERTIBLE MATRICES

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1. Introduction

Let $A = [a_{ij}]$ be any real matrix of order n. The *permanent* of A is defined by

$$perA = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where S_n stands for the symmetric group on $\{1, 2, \cdots, n\}$. Even though the permanent and the determinant look very much alike as matrix function, most problems involving permanents are considerably more difficult than the corresponding problems for determinants. It would therefore be of interest to find a transformation that would convert permanents into determinants. An $n \times n$-matrix A is called *convertible* if there is an (1,-1)-matrix H such that $perA = det(H \circ A)$ where $H \circ A$ denotes the Hadamard(entrywise) product of H and A. Such a matrix H is called a converter of A.

For matrices A, B of the same size, A is said to be *permutation equivalent* to B, denoted by $A \sim B$, if there are permutation matrices P, Q such that $PAQ = B$. Let $T_n = [t_{ij}]$ denote the $n \times n$ (0,1)-matrix such that $t_{ij} = 0$ if and only if $j > i + 1$, and for a matrix or a vector A, let $\nu(A)(\mu(A))$ denote the number of positive(negative) entries of A. Gibson[1] proved that for any $n \times n$ convertible (0,1)-matrix A with $perA > 0$, $\nu(A) \leq \frac{1}{2}(n^2 + 3n - 2)$ with equality if and only if $A \sim T_n$. The structure of $n \times n$ convertible (0,1)-matrices with exactly $\frac{1}{2}(n^2 + 3n - 2)$ 1's was completely determined by Kräuter and Seifter [6]. An $n \times n$ convertible (0,1)-matrix is called *extremal* if replacing any one of zero entries of it by 1 yields a nonconvertible matrix. The matrix T_n is, for example, an extremal matrix. An $n \times n$ matrix is called *fully indecomposable* if it does not contain a $t \times (n - t)$ zero submatrix.

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In this paper, we investigate the structure of a special class of convertible matrices. This class contains the set which is studied by Kräuter and Seifter. For positive integers k and n with $k \leq n$, let $Q_{k,n}$ denote the set of all strictly increasing k-sequences from $1, 2, \cdots, n$. For an $n \times n$ matrix A and for $\alpha, \beta \in Q_{k,n}$, let $A(\alpha|\beta)$ denote the submatrix obtained from A by deleting rows α and columns β. Let $\mathbf{e}_j$ denote the j-th column of the identity matrix of suitable order.

2. Main results

Let $\mathcal{R}_n$, a set of $n \times n$ convertible (0,1)-matrices, be inductively defined as follows: Let $\mathcal{R}_2$ consist of T_2 only. For $n \geq 3$, let $A = [a_{ij}] \in \mathcal{R}_n$ if and only if A is an $n \times n$ extremal convertible (0,1)-matrix such that $\nu((a_{11}, \cdots, a_{1n})) = 2$ and $A(1|j) \in \mathcal{R}_{n-1}$ whenever $a_{1j} = 1$. Similarly, let $\mathcal{C}_2 = \mathcal{R}_2$. For $n \geq 3$, let $A = [a_{ij}] \in \mathcal{C}_n$ if and only if A is an $n \times n$ extremal convertible (0,1)-matrix such that $\nu((a_{11}, \cdots, a_{n1})) = 2$ and $A(i|1) \in \mathcal{C}_{n-1}$ whenever $a_{i1} = 1$. It is easy to show that A is fully indecomposable and $perA = 2^{n-1}$ for any $A \in \mathcal{R}_n(\mathcal{C}_n)$.

We rewrite the following results in [2,4] as lemmas before we state our first result.

LEMMA A ([2]). *A square submatrix of a convertible (0,1)-matrix is convertible if its complementary submatrix has a positive permanent.*

LEMMA B ([4]). *Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ be an $n \times n$ (0,1)-matrix and $\mathbf{b}$ be an n-vector of 0's and 1's. For $k \in \{1, 2, \cdots, n\}$, let*

$$B = \begin{pmatrix} 1 & \mathbf{e}_k^T \\ \mathbf{b} & A \end{pmatrix}.$$

If B is fully indecomposable and extremal convertible, then so is A and $\mathbf{b} = \mathbf{a}_k$.

LEMMA 1. *Let $A = [a_{ij}] \in \mathcal{R}_n$ with $a_{1j} = a_{1k} = 1 (i \neq j)$. Let H be a converter of A such that $A^* = [a^*_{ij}] = H \circ A$ satisfies $\mu((a^*_{1j}, \cdots, a^*_{nj})) = 0$. Then $\nu((a^*_{2k}, \cdots, a^*_{nk})) = 0$ or $\mu((a^*_{2k}, \cdots, , a^*_{nk})) = 0$ according to $a^*_{1k} = 1$ or -1. A similar statement holds for $A \in \mathcal{C}_n$.*

Proof. By Lemma B, the j-th and k-th rows of A are equal. Since $A(1|j)$ is fully indecomposable, $perA(1, l|j, k) > 0$ for any $l \in \{2, \cdots, n\}$. By lemma A, $a^*_{lk} = -a^*_{1k}$ for any l with $a^*_{lj} \neq 0$.

Let $A = [a_{ij}] \in \mathcal{R}_n(\mathcal{C}_n)$. Without loss of generality we may assume that $a_{11} = 1$ (otherwise we permute rows or columns of A). We shall use this fact in all our proofs without mentioning it again. For convenience's sake, let $min\{\mu(A^{\#})\}$ denote the minimal integer in $\{\mu(H \circ A)|H$: a converter of $A\}$.

THEOREM 1. *Let* $A = [a_{ij}] \in \mathcal{R}_n(\mathcal{C}_n)$ *and* $min\{\mu(A^{\#}(1|1))\} = p$. *Then* $\mu(H \circ A) > p$ *for any converter* H *of* A.

Proof. Let $A = [a_{ij}] \in \mathcal{R}_n$. Without loss of generality, we may assume that $a_{11} = a_{1k} = 1$, $a_{21} = \cdots = a_{j-11} = 0$ and $a_{j1} = \cdots = a_{n1} = 1$. Notice that if H is a converter of $A = [a_{ij}]$, then $H(i|j)$ is a converter of $A(i|j)$ for any (i,j) with $a_{ij} = 1$. Hence we have $\mu(H \circ A) \geq p$ for any converter H of A. Suppose that $\mu(H \circ A) = p$ for some converter H of A. Write $A^* = [a^*_{ij}] = H \circ A$. Then $a^*_{11} = a^*_{1k} = a^*_{j1} = \cdots = a^*_{n1} = 1$. By Lemma 1, $a^*_{jk} = \cdots = a^*_{jn} = -1$. Since $A(1|1)$ is fully indecomposable, $j \leq n-1$. Multiplying -1 to the first row and k-th column of A^*, we have another converter H' of A such that $\mu(H' \circ A) = \mu((H' \circ A)(1|1)) + 1 = p - (n-j+1) + 1 \leq p - 1$. This is a contradiction.

COROLLARY. *Let* $A = [a_{ij}] \in \mathcal{R}_n(\mathcal{C}_n)$. *Then* $\mu(H \circ A) \geq n-1$ *for any converter* H *of* A.

In the proof of Theorem 1, we proved a little more than the statement of the theorem. What we have actually shown is that the theorem holds for fully indecomposable, extremal convertible (0,1)-matrix whose first row or first column has only two nonzero entries.

Let $U_2 = T_2$ and let

$$U_n = \begin{pmatrix} 1 & \mathbf{a} \\ \mathbf{b} & U_{n-1} \end{pmatrix}$$

for $n \geq 3$ where

$$\mathbf{a} = \left(1, \frac{1+(-1)^n}{2}, 0, \cdots, 0\right), \ \mathbf{b} = \left(1, \frac{1-(-1)^n}{2}, 0, \cdots, 0\right)^T.$$

COROLLARY. $min\{\mu(T_n^{\#})\} = min\{\mu(U_n^{\#})\} = n-1$.

Proof. Let H be the $n \times n$ (1,-1)-matrix such that $h_{ij} = -1$ if and only if $(i,j) \in \{(1,2),(2,3),\cdots,(n-1,n)\}$. Then H is a converter of T_n and also U_n such that $\mu(H \circ T_n) = \mu(H \circ U_n) = n-1$.

We are now ready to show characterizations of $A \in \mathcal{R}_n(\mathcal{C}_n)$ with $min\{\mu(A^{\#})\} = n-1$.

THEOREM 2. *Let $A \in \mathcal{R}_n(\mathcal{C}_n)$ with $\mu(H \circ A) = n-1$ for some converter H of A. Then every row and column of $H \circ A$ contain at most one negative element.*

Proof. Without loss of generality, we may assume that

$$A = [a_{ij}] = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & & * & \\ 1 & 1 & & & \\ \vdots & \vdots & & & \\ 1 & 1 & & & \end{pmatrix}.$$

Let $k(>1)$ be the first integer such that $a_{k1} = 1$. We will prove by induction on n. For $n = 1$, there is nothing to prove. Assume that the result holds for $m < n$. Let H be a converetr of A such that $\mu(H \circ A) = n-1$. Write $A^* = [a^*_{ij}] = H \circ A$. If $\mu(A^*(1|1)) = n-1$, then $\mu((a^*_{11}, a^*_{k1}, \cdots, a^*_{n1}, a^*_{12})) = 0$. Thus $a^*_{k2} = \cdots = a^*_{n2} = -1$. Since $A(1|1)$ is fully indecomposable, $k \leq n-1$. Multiplying -1 to the second column and the first row of A^*, we have a converter H' of A such that $\mu(H' \circ A) = k-1 < n-1$, which is impossible. Therefore $\mu(A^*(1|1)) = n-2$. By hypothesis, every row and every column of $A^*(1|1)$ contains at most one negative element. Since $\mu(A^*(1|1)) = n-2$, we have $\mu((a^*_{11}, a^*_{k1}, \cdots, a^*_{n1}, a^*_{12})) = 1$. If $a^*_{11} = -1$, then we have the result. If $a^*_{12} = -1$, then we have $a^*_{k2} = \cdots = a^*_{n2} = 1$ by Lemma 1. Thus we have the result. If $a^*_{j1} = -1$ for some j with $k \leq j \leq n$, without loss of generality, we may assume that $a^*_{k1} = -1$. Then we have $a^*_{k2} = 1$ and $a^*_{k+1,2} = \cdots = a^*_{n2} = -1$. This implies that $k \geq n-1$ by hypothesis. Since A is fully indecomposable, $k = n-1$. Thus $a^*_{n-1,2} = 1$ and $a^*_{n2} = -1$. Hence $\mu((a^*_{n3}, \cdots, a^*_{nn})) = 0$. This implies that $\mu((a^*_{n-1,3}, \cdots, a^*_{n-1,n})) = 0$ by Lemma 1. Thus we have the result.

Let $T_{n-1} = [\mathbf{t}_1, \cdots, \mathbf{t}_{n-1}] = [\mathbf{x}_1, \cdots, \mathbf{x}_{n-1}]^T$ and for $k = 1, 2, \cdots, n-1$, let

$$V_{n,k} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \mathbf{t}_k & \mathbf{t}_1 & \cdots & \mathbf{t}_{k-1} & \mathbf{t}_k & \mathbf{t}_{k+1} & \cdots & \mathbf{t}_{n-1} \end{pmatrix}.$$

and

$$W_{n,k} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \mathbf{x}_k & \mathbf{x}_1 & \cdots & \mathbf{x}_{k-1} & \mathbf{x}_k & \mathbf{x}_{k+1} & \cdots & \mathbf{x}_{n-1} \end{pmatrix}^T.$$

Notice that $V_{n,k}(W_{n,k} \in \mathcal{R}_n(\mathcal{C}_n)$ for each $k = 1, 2, \cdots, n-1$.

THEOREM 3.

$$min\{\mu(V^{\#}_{n,k})\} = \begin{cases} n-1 & \text{if } k = 1,2,n-1 \\ n & \text{otherwise.} \end{cases}$$

Proof. Notice that $min\{\mu(T^{\#}_{n-1})\} = n-2$. Hence $\mu(H \circ V_{n,k}) \geq n-1$ for any converter H of $V_{n,k}$. Since $V_{n,1} \sim V_{n,2} \sim T_n$, the statement holds for $k = 1,2$. For $k = n-1$, let $H = [h_{ij}]$ be the $n \times n$ (1,-1)-matrix such that $h_{ij} = -1$ if and only if $(i,j) \in \{(n-1,1),(2,2),(3,4),\cdots,(n-2,n-1),(n,n)\}$. Then H is a converter of $V_{n,n-1}$ with $\mu(H \circ V_{n,n-1}) = n-1$. For $k \neq 1,2,n-1$, let $H = [h_{ij}]$ be the $n \times n$ (1,-1)-matrix such that $h_{ij} = -1$ if and only if $(i,j) \in \{(1,k+1),(k,1),(2,3),\cdots,(n-1,n)\}$. Then H is a converter of $V_{n,k}$ and $\mu(H \circ V_{n,k}) = n$. Suppose that $min\{\mu(V^{\#}_{n,k})\} < n$. Then $min\{\mu(V^{\#}_{n,k})\} = n-1$. Let H be a converter of $V_{n,k}$ such that $\mu(H \circ V_{n,k}) = n-1$. Write $V^{*}_{n,k} = [v^{*}_{ij}] = H \circ V_{n,k}$. Then $\mu(v^{*}_{11}, v^{*}_{1k+1}, v^{*}_{k1}, \cdots, v^{*}_{n1}) \leq 1$.

Case $1 : \mu(v^{*}_{11}, v^{*}_{1k+1}, v^{*}_{k1}, \cdots, v^{*}_{n1}) = 0$.

By Lemma 1, we have $v^{*}_{kk+1} = \cdots = v^{*}_{nk+1} = -1$. Since $\mu(V^{*}_{n,k}) = n-1$, by Theorem 2, $k \geq n$, which is impossible.

Case $2 : \mu(v^{*}_{11}, v^{*}_{1k+1}, v^{*}_{k1}, \cdots, v^{*}_{n1}) = 1$.

Without loss of generality, we may assume that $v^{*}_{1k+1} = -1$ or $v^{*}_{k1} = -1$. If $v^{*}_{k1} = -1$, then $v^{*}_{kk+1} = 1$ and $v^{*}_{k+1,k+1} = \cdots = v^{*}_{n,k+1} = -1$. By Theorem 2, $k+1 \geq n$, which is impossible. If $v^{*}_{1k+1} = -1$, then k-th column of $V^{*}_{n,k}(1|1)$ is nonnegative. Also either the first row or second row of $V^{*}_{n,k}(1|1)$ should be nonnegative. This contradicts to Theorem 2.

Theorem 3 also holds for $W^{\#}_{n,k}$. That is,

$$min\{\mu(W^{\#}_{n,k})\} = \begin{cases} n-1 & \text{if } k = 1,n-1,n \\ n & \text{otherwise.} \end{cases}$$

THEOREM 4. *Let $A \in \mathcal{R}_n(C_n)$ with $min\{\mu(A^{\#})\} = n-1$. Then $A \sim T_n$ or $A \sim V_{n,n-1}(W_{n,1})$.*

Proof. We will prove by induction on n. There is nothing to prove for $n \leq 4$ by Theorem 3. Assume that the statement holds for $m < n$. Let $A \in \mathcal{R}_n$ with $min\{\mu(A^{\#})\} = n-1$. Since $A(1|1)$ satisfies the condition above, $A(1|1) \sim T_{n-1}$ or $A(1|1) \sim V_{n-1,n-2}$. If $A(1|1) \sim T_{n-1}$, by Theorem 3,

$A \sim T_n$ or $A \sim V_{n,n-1}$. If $A(1|1) \sim V_{n-1,n-2}$, then $A \sim B$ where $B = [b_{ij}]$ is of the form

$$B = \begin{pmatrix} 1 & & b_{1j} & \\ & 1 & & 1 \\ & & & \\ & * & T_{n-2} & \\ & & & \end{pmatrix}.$$

If $n = 5$, it is easy to show that $B \sim V_{5,4}$. Thus we may assume that $n \geq 6$. It is sufficient to show that if $b_{1j} = 1$, then $j = 3$ or 4. For, permuting the first two rows and the first two columns(and 3rd, 4th columns if necessary), we obtain $A \sim V_{n,n-1}$. Suppose that $j \neq 3, 4$. Let H be a converter of B such that $\mu(H \circ B) = n - 1$. Write $B^* = [b^*_{ij}] = H \circ B$. It is easy to prove that $\mu(B^*(1,2|1,2)) = n - 3$. By Theorem 2, every row and every column of $B^*(1,2|1,2)$ contains at most one negative element. Thus without loss of generality, we may assume that $b^*_{1j} = b^*_{34} = -1$ because of $n \geq 6$. Then the first column of $B^*(1,2|1,2)$ has no negative elements. If $j = 2$, then $b^*_{2n} = -1$ and hence the last column of $B^*(1,2|1,2)$ has no negative elements. This is a contradiction by Theorem 2. If $j \neq 2$, then $(j-2)$-th column of $B^*(1,2|1,2)$ has no negative elements. This is also a contradiction. Thus $A \sim T_n$ or $A \sim V_{n,n-1}$.

THEOREM 5. *Let $A \in \mathcal{R}_n(\mathcal{C}_n)$ with $min\{\mu(A^{\#})\} = n - 1$. Then for any converter H of A, we can obtain a converter H' of A from $H \circ A$ with $\mu(H' \circ A) = n - 1$ by multiplying -1 to some rows and columns of $H \circ A$.*

Proof. By Theorem 4, we may assume that $A = T_n$ or $A = V_{n,n-1}$.

Case 1 : $A = T_n$.

Given converter H_α of T_n, write $T^\alpha_n = H_\alpha \circ T_n = [t^\alpha_{ij}]$. Let H_0 be a converter of T_n. Multiply -1 to j-th row of T^0_n whenever $t^0_{j1} = -1$ and multiply -1 to 2nd column of T^0_n if $t^0_{12} = 1$. If the number of lines which are multiplied with -1 is odd, then also multiply -1 to the last column of T^0_n. Anyway, we have a converter H_1 of T_n such that the first row of $T^1_n(1|1)$ is positive. If $t^1_{23} = 1$, then multiply -1 to the 3rd column and last column of T^1_n. Then we have a converter H_2 of T_n such that the first row of $T^2_n(1,2|1,2)$ is positive. Continuing this process, we have a converter H_j of T_n such that the first row of $T^j_n(1,2,\cdots,j|1,2,\cdots,j)$ is positive for

$j = 1, 2, \cdots, n-1$. Thus we have

$$H_{n-1} \circ T_n = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & -1 \\ 1 & \cdots & \cdots & \cdots & 1 \end{pmatrix}$$

with $\mu(H_{n-1} \circ T_n) = n - 1$.

Case 2 : $A = V_{n,n-1}$.

Since $V_{n,n-1}(1|1) = T_{n-1}$, by case 1, we have a converter H of $V_{n,n-1}$ such that

$$H \circ V_{n,n-1} = [v^*_{ij}] = \begin{pmatrix} v^*_{11} & 0 & \cdots & \cdots & 0 & v^*_{1n} \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \vdots & & \ddots & \ddots & 0 \\ v^*_{n-11} & \vdots & & & \ddots & -1 \\ v^*_{n1} & 1 & \cdots & \cdots & \cdots & 1 \end{pmatrix}.$$

Then $v^*_{11} = 1$. If $v^*_{1n} = 1$, $v^*_{n-1,1} = 1$ and $v^*_{n1} = -1$. Thus we have $\mu(H_1 \circ V_{n,n-1}) = n - 1$. If $v^*_{1n} = -1$, then $v^*_{n-1,1} = -1$ and $v^*_{n1} = 1$. Multiplying -1 to the 2nd row and last column of $V^*_{n,n-1}$, we have a converter H_2 of $V_{n,n-1}$ such that $\mu(H_2 \circ V_{n,n-1}) = n - 1$. Notice that we can get $H_1 \circ V_{n,n-1}$ from $H_2 \circ V_{n,n-1}$ by multiplying -1 to some rows and columns of $H_2 \circ V_{n,n-1}$.

As an immediate consequences, we obtain the following result due to Kräuter and Seifter[6].

COROLLARY. *For $n - 1 \le i \le \binom{n+1}{2}$, there exists a converter H of A such that $\mu(H \circ T_n) = i$.*

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