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COMMON FIXED POINTS ON UNIFORMLY CONVEX BANACH SPACES

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Let K be a nonempty subset of a uniformaly convex Banach space X. K is said to be a T-regular set if $T : K \to K$ and $\frac{1}{2}(x + Tx) \in K$ for $x \in K$. Recently Veeramani [2] has proved that every convex set invariant under a map T is a T- regular set and a T-regular set need not be a convex set.

Let $\mathcal{T}$ be a family of maps from K into itself. K is called $\mathcal{T}$-regular provided that each $T \in \mathcal{T}$ is T-regular. The main purpose in this paper is to give some properties of the set of common fixed points of a family of commuting maps over a nonempty weakly compact $\mathcal{T}$-regular subset of a uniformly convex Banach space. Our results extend properly the main results of Browder [1] and Veeramani [2]. Throughout this paper $\mathbf{R}$ is the set of all real numbers, $\delta(K)$ and $F_{\mathcal{T}}$ denote the diameter of K and the set of common fixed points of $\mathcal{T}$. For $x \in X$, define $\delta(x, K) = \sup_{y \in K} \|x - y\|$. Let T be a self map of K, F_T denote the set of fixed points of T.

LEMMA 1 [2]. *Let $\{A_\alpha\}_{\alpha \in I}$ be a collection of T-regular subsets of a vector space X. Then $\cap_{\alpha \in I} A_\alpha$ is a T-regular set.*

LEMMA 2 [2]. *Let C be a bounded T-regular subset of a uniformly convex Banach space X. Then either $Tx = x$ for all $x \in C$ or there exists $w \in C$ such that $\delta(w, C) < \delta(C)$.*

The following result improves Theorem 1.1 of Veeramani[2].

LEMMA 3. *Let K be a nonempty weakly compact T-regular subset of a uniformly convex Banach space X. Further for each weakly closed T-regular subset C of K with $\delta(C) > 0$, there exists some $\beta \in (0, 1)$ such that*

$$(1) \qquad \|Tx - Ty\| \le \max\{\|x - y\|, \min\{\|x - Ty\|, \|y - Tx\|\}, \beta\delta(C)\}$$

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for all $x, y \in C$. *Then* $F_T \neq \phi$.

Proof. Let H be the collection of all nonempty weakly closed, T- regular subset of K. By Lemma 1 and Zorn's lemma H has a minimal element, say, C. We assert that $Tx = x$ for all $x \in C$. Otherwise there exists $x \in C$ such that $x \neq Tx$. By Lemma 2 there exists $w \in C$ and $a \in (0,1)$ such that $\delta(w, C) \leq \alpha\delta(C)$. It follows from hypothesis that there exists $\beta \in (0,1)$ such that for each $y \in C$,

$$\begin{aligned}\|Tw - Ty\| &\leq \max\{\|w - y\|, \min\{\|w - Ty\|, \|y - Tw\|\}, \beta\delta(C)\} \\ &\leq \max\{\delta(w, C), \min\{\delta(w, C), \|y - Tw\|\}, \beta\delta(C)\} \\ &= \max\{\delta(w, C), \beta\delta(C)\} \leq t\delta(C)\end{aligned}$$

where $t = \max\{\alpha, \beta\}$. Let $E = \{x \in K : \delta(x, C) \leq t\delta(C)\}$ and $D = E \cap C$. Then $w \in D \neq \phi$. It is easily seen that E is closed and convex. Hence D is weakly closed.

Let $x \in D$. Then for any $y \in C$ we have by (1)

$$\begin{aligned}\|Tw - Ty\| &\leq \max\{\|x - y\|, \min\{\|x - Ty\|, \|y - Tx\|\}, \beta\delta(C)\} \\ &\leq \max\{t\delta(C), \min\{t\delta(C), \|y - Tx\|\}, \beta\delta(C)\} = t\delta(C).\end{aligned}$$

Hence $TC \subset V = \{z \in K : \|z - Tx\| \leq t\delta(C)\}$. Clearly $C \cap V \supset TC \neq \phi$. Let $y \in C \cap V$. Then $Ty \in C$ and $\|Ty - Tx\| \leq t\delta(C)$; i.e., $Ty \in V$. Consequently $C \cap V$ is T- invariant. Since C is a T-regular set and V is a convex set, $C \cap V$ is a T-regular set. Note that V is closed and convex. Then $C \cap V$ is weakly closed. The minimality of C yields $C = C \cap V$. This implies $C \subset V$. Therefore for any $y \in C \subset V$, $\|Tx - Ty\| \leq t\delta(C)$. Consequently $\delta(Tx, C) \leq t\delta(C)$; i.e., $Tx \in E$. Hence $Tx \in E \cap C = D$, which implies $TD \subset D$. By the convexity of E and the T-regularity of C, D is a T-regular set. By the minimality of C, $C = D = E \cap V$. This implies $C \subset E$ and

$$\delta(C) = \sup_{x,y \in C} \|x - y\| \leq \sup_{x \in C} \delta(x, C) \leq \sup_{x \in E} \delta(x, C) \leq t\delta(C) < \delta(C)$$

which is impossible. Hence $Tx = x$ for all $x \in C$. Consequently $F_T \supset C \neq \phi$. This completes the proof.

Our main result is as follows:

Theorem 1. *Let K be a nonempty weakly compact $\mathcal{T}$-regular subset of a uniformly convex Banach space X and $\mathcal{T} : K \to K$ be a family of commuting maps satisfying*

$$(2) \qquad \|Tx - Ty\| \leq \max\{\|x-y\|, \min\{\|x - Ty\|, \|y - Tx\|\}$$

for $T \in \mathcal{T}$ and $x, y \in K$. Then $F_{\mathcal{T}}$ is nonempty closed convex and, in particular, weakly compact.

Proof. Clearly $F_{\mathcal{T}} = \cap_{T \in \mathcal{T}} F_T$. Let $T \in \mathcal{T}$. It follows from Lemma 3 that $F_T \neq \phi$. Let $\{x_n\} \subset F_T$ and $x_n \to x$ as $n \to \infty$. Using (2),

$$\begin{aligned}\|x - Tx\| &\leq \|x_n - x\| + \|Tx_n - T_x\| \\ &\leq \|x_n - x\| + \max\{\|x_n - x\|, \min\{\|x_n - Tx\|, \|x - x_n\|\}\} \\ &= 2\|x_n - x\|.\end{aligned}$$

It is easy to prove that $x = Tx \in F_T$. Hence F_T is closed. Let x_1, $x_2 =\in F_T$ and $x = \frac{1}{2}(x_1 + x_2)$. Then

$$\|x_i - Tx\| \leq \max\{\|x_i - x\|, \min\{\|x_i - Tx\|, \|x_i - x\|\}\} = \frac{1}{2}\|x_1 - x_2\|.$$

Conesequently

$$\|x - Tx\| \leq \frac{1}{2}(\|x_1 - Tx\| + \|x_2 - Tx\|) \leq \frac{1}{2}\|x_1 - x_2\|$$

and

$$\|x_1 - x_2\| \leq \|x_1 - Tx\| + \|Tx - x_2\| \leq \|x_1 - x_2\|.$$

In view of strict convexity of the norm, $x_i - Tx$ and hence, Tx must lie on the segment joining x_1 and x_2. The inequalities $\|x_i - Tx\| \leq \frac{1}{2}\ \|x_1 - x_2\|$ implies that fx is the midpoint. Therefore $x = Tx$ and F_T is convex. It follows that F_T is closed convex. As closed convex subset of the weakly compact set K, F_T is weakly compact. In order to show that $F_{\mathcal{T}} \neq \phi$ it suffices to show that $\{F_T : T \in \mathcal{T}\}$ has the finite intersection property.

We make the inductive assumption that each n members of $\mathcal{T}$ have a common fixed point in K. Let T_1, T_2, $\cdots$, $T_{n+1} \in \mathcal{T}$. Then $\cap_{i=1}^n F_{T_i}$ is nonempty weakly compact. It follows from the commutatively of $\mathcal{T}$ that $T_{n+1} \cap_{i=1}^n F_{T_i} \subset \cap_{i=1}^n F_{T_i}$. By Lemma 1, T_{n+1} has a fixed point $y \in K$. The strict convexity of the norm together with the weak compactness of

$\cap_{i=1}^n F_{T_i}$ enable us to find a unique point $x \in \cap_{i=1}^n F_{T_i}$ nearest to y. By(2) we get

$$\|T_{n+1}x - T_{n+1}y\| \le \max\{\|x-y\|, \min\{\|x-y\|, \|y - T_{n+1}x\|\}\} = \|x-y\|.$$

Note that $T_{n+1}x \in \cap_{i=1}^n F_{T_i}$. Hence $\|T_{n+1}x - y\| = \|x-y\|$. Consequently $x = T_{n+1}x \in \cap_{i=1}^n F_{T_i}$; i.e., $\cap_{i=1}^{n+1} F_{T_i} \neq \phi$. This completes the proof.

Now we give some consequences of Theorem 1.

COROLLARY 1. *Let K be a nonempty weakly compact $\mathcal{T}$-regular subset of a uniformly convex Banach space X and $\mathcal{T}$ be a commutative family of nonexpansive maps of K into itself. Then $F_{\mathcal{T}}$ is nonempty closed convex.*

COROLLARY 2. *Let K be a nonempty weakly compact T-regular subset of a uniformly convex Banach space X, $T : K \to K$ satisfy (2) for $x, y \in K$. Then F_T is nonempty closed convex.*

Corollary 1 extends Corollary 1.2 of Veeramani [2]. The following example shows that Corollary 2 is a proper generalization of Veeramani's Corollary 1.2 and Browder's theorem.

EXAMPLE 1. Let $X = \mathbb{R}$ with the usual norm and $K = [0,1]$. Define a map $T : K \to K$ by

$$Tx = \begin{cases} \dfrac{x}{3} & \text{if } x \in [0,1] \text{ and } X \text{ is rational} \\ 0 & \text{if } x \in [0,1] \text{ and } X \text{ is irrational} \end{cases}$$

Then all the conditions of Corollary 2 are satisfied. But Veeramani's Corollary 1.2 and Browder's theorem are not applicable since T does not satisfy $\|Tx - Ty\| \le \|x-y\|$ for $x = 1$ and $y = \frac{\sqrt{2}}{2}$.

THEOREM 2. *Let K be a nonempty weakly compact $\mathcal{T}$-regular subset of a uniformly convex Banach space X and $\mathcal{T} : K \to K$ be a family of commuting maps satisfying (2) for $T \in \mathcal{T}$ and $x, y \in K$. Suppose that $h : X \to \mathbb{R}$ satisfies*

(i) h is lower semicontinuous convex function;

(ii) $hTx \le hx$ for $T \in \mathcal{T}$ and $x \in K$.

Then there exists $x_0 \in K$ such that $hx_0 = \inf\{hx : x \in K\}$ and $Tx_0 = x_0$ for all $T \in \mathcal{T}$.

Proof. Let $A = \{x \in K : hx = \inf_{y \in K} hy\}$. As in the proof of Theorem 2.1 [2], we conclude that $A = \{x \in X : hx \le \inf_{y \in K} hy\} \cap K \neq \phi$ and A is weakly compact. It follows from (ii) that $TA \subset A$ for each $T \in \mathcal{T}$. Let $x \in A$ and $T \in \mathcal{T}$. By (i) we have

$$h(\frac{x+Tx}{2}) \le \frac{hx}{2} + \frac{hTx}{2} \le hx = \inf_{y\in K} hy$$

which implies $h(\frac{x+Tx}{2}) = \inf_{y\in K} hy$. Consequently $\frac{x+Tx}{2} \in A$ and A is $\mathcal{T}$-regular. By Theorem 1 there exists $x_0 \in A$ such that $Tx_0 = x_0$ for each $T \in \mathcal{T}$. This completes the proof.

COROLLARY 3. *Let K be a nonempty weakly compact $\mathcal{T}$-regular subset of a uniformly convex Banach space X and $\mathcal{T} : K \to K$ be a family of commuting maps satisfying (2) for $T \in \mathcal{T}$ and $x, y \in K$. Suppose $y_0 \in X \backslash K$ and $\|Tx - y_0\| \le \|x - y_0\|$ for $T \in \mathcal{T}$ and $x \in K$. Then $\mathcal{T}$ has a common fixed point $x_0 \in K$ which is a best approximation to y_0 from K.*

Proof. Define a function $h : X \to \mathbb{R}$ by $hx = \|x - y_0\|$. Then h is a continuous convex function. For $x \in K$ and $T \in \mathcal{T}$, we have

$$hTx = \|Tx - y_0\| \le \|x - y_0\| = hx.$$

Hence by Theorem 2 there exists $x_0 \in K$ such that $x_0 = Tx_0$ for all $T \in \mathcal{T}$ and

$$\|x_0 - y_0\| = hx_0 = \inf_{x\in K} \|x - y_0\|.$$

This completes the proof.

The following example reveals that Theorem 2 and Corollary 3 extend properly Theorem 2.1 and Corollary 2.2 of Veeramani.

EXAMPLE 2. Let X, K and T be as in Example 1. Let $y_0 = -1$, $\mathcal{T} = \{T\}$. Define $h : X \to \mathbb{R}$ by $hx = \|x - y_0\|$. Then the assumptions of Theorem 2 and Corollary 3 are satisfied but Theorem 2.1 and Corollary 2.2 of Veeramani are not applicable since T is not nonexpansive.

References

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