Honam Mathematical J. 16(1994), No. 1,pp. 17-22

ON A CERTAIN CLASS OF EUCLIDEAN HYPERSURFACES

DONG-SOO KIM, SEUNG-HO AHN AND HEI-SUN CHUNG

Dept. of Mathematics, Chonnam National University, Kwangju 500-757, Korea.

1. Introduction

Let $x : M^n \to R^m$ be an isometric immersion of a connected manifold M into the Euclidean space R^m and Δ its Laplacian defined by -div o grad. This Laplacian can be extended in a natural way to vector valued functions on M and we have the useful formula ([1]):

$$\Delta x = -nH, \tag{1.1}$$

where H is the mean curvature vector field of M. It follows from Takahashi's theorem ([5]) that the solutions to the equation

$$\Delta x = \lambda x + b, \quad \lambda \in R, \quad b \in R^m, \tag{1.2}$$

are precisely those submanifolds which are either minimal in R^m (for $\lambda = 0$) or minimal in the hypersphere $S^{m-1}_{x_0}(r)$ (for $\lambda \neq 0$, in this case, we have $r = \sqrt{n}/\sqrt{\lambda}$ and $x_0 = -b/\lambda$).

Recently, generalizing the Takahashi's condition (1.2), many authors ([2, 3, 4]) studied the hypersurfaces in R^{m+1} satisfying the condition

$$\Delta x = Ax + b, \quad A \in M_{n+1}(R), \quad b \in R^{n+1}. \tag{1.3}$$

And they proved that a connected complete hypersurface of R^{n+1} which satisfies (1.3) is a minimal hypersurface, a hypersphere or a generalized circular cylinder.

On the other hand, a circular cylinder in R^3 over a circle $S^1_{x_0}(r)$ with axis u satisfies the following condition:

$$\Delta x = \frac{1}{r^2}x - \frac{1}{r^2}x_0 - \frac{1}{r^2} < x - x_0, u > u. \tag{1.4}$$

Received April 29, 1994.

This work was partially supported by BSRI-94-1425, Ministry of Education, 1994.

Thus it is worth while to classify the hypersurfaces in R^{n+1} satisfying

(1.5) $\Delta x = \lambda x + b + g(x)u, \quad \lambda \in R, \quad b \in R^{n+1}, \quad |u| = 1$ and $g \in C^\infty(M)$.

In this paper, we prove that the complete minimal hypersurfaces in R^{n+1} are the only complete ones which satisfy the condition (1.5) with $\lambda = 0$ and we classify the complete surfaces in R^3 satisfying (1.5) with $\lambda \neq 0$. As a corollary, we prove that the spheres are the only compact in R^3 satisfying (1.5).

Note that for $\lambda \neq 0$, the condition (1.5) is invariant under the congruences in R^{n+1}, and that for $\lambda = 0$, (1.5) is invariant under the rotations in R^{n+1}.

2. Preliminaries

Suppose that an isometric immersion $x : M^n \to R^{n+1}$ satisfies the condition (1.5). Let G and α denote a unit normal vector field of M and the mean curvature of M w.r.t. G. Since $\Delta x = -nH = -n\alpha G$, we have

$$(2.1) \qquad -n\alpha G = \lambda x + b + gu.$$

We may choose a local orthonormal frame $e_1, \cdots, e_n$ of M such that $Ae_i = \mu_i e_i$, $i = 1, \cdots, n$, where A is the shape operator of M w.r.t. G. Then we have $n\alpha = \mu_1 + \cdots + \mu_n$. And by differentiating (2.1), we obtain

$$(2.2) \qquad e_i(g)u = (n\alpha\mu_i - \lambda)e_i - ne_i(\alpha)G, \quad i = 1, \cdots, n.$$

Suppose that $U_1 = \{p \in M | e_1 g(p) \neq 0\}$ is not empty. Then on U_1, we have

$$(2.3) \qquad e_i(g) = e_i(\alpha) = n\alpha\mu_i - \lambda = 0, \quad i = 2, \cdots, n,$$

$$(2.4) \qquad u = \beta e_1 + \gamma G, \quad \beta = (n\alpha\mu_1 - \lambda)/e_1(g), \quad \gamma = -ne_1(\alpha)/e_1(g).$$

Differentiating (2.4) w.r.t. e_i implies the following:

$$(2.5) \qquad \bar{\nabla}_{e_1} e_1 = \mu_1 G, \quad e_1(\beta) = \gamma\mu_1, \quad e_1(\gamma) = -\beta\mu_1,$$

$$(2.6) \qquad \beta\bar{\nabla}_{e_1} e_i = \gamma\mu_i e_i, \quad e_i(\beta) = e_i(\gamma) = 0, \quad i = 2, \cdots, n,$$

where $\bar{\nabla}$ denotes the flat connection of R^{n+1}. We put $v = -\gamma e_1 + \beta G$ and

$$U_2 = \{p \in U_1 | \alpha(p)(e_1\alpha)(p) \neq 0\}, \quad U_3 = \{p \in U_2 | \beta(p) \neq 0\}.$$

Suppose that $U_3 \neq \phi$, then on U_3 we have the following:

$$\bar{\nabla}_{e_1}(e_1 \wedge G) = 0, \tag{2.7}$$

$$\bar{\nabla}_{e_1} v = 0, \quad \bar{\nabla}_{e_i} v = -\frac{\mu_i}{\beta} e_i = \frac{-\lambda}{n\alpha\beta} e_i, \quad i = 2, \cdots, n. \tag{2.8}$$

Note that $\{v, e_2, \cdots, e_n\}$ generates the fixed hyperplane Π in R^{n+1} which is orthogonal to u. Let $c(s)$ be an integral curve of e_1. Then (2.7) shows that $c(s)$ is a plane curve in a plane generated by $\{u, v\}$. From (2.6) we see that $\{e_2, \cdots, e_n\}$ defines an integrable distribution Δ on M. Let $S(s)$ be a leaf of Δ through $c(s)$. Then $S(s)$ is a hypersurface in Π in which v is a unit normal vector field of $S(s)$. Hence (2.8) shows that $S(s)$ is an open part of either a hyperplane (for $\lambda = 0$) in Π or a hypersphere (for $\lambda \neq 0$) in Π. Thus for $\lambda = 0$, U_3 is an open part of a cylindrical hypersurface in R^{n+1} over a curve $c(s)$ in the plane generated by $\{u, v\}$. And for $\lambda \neq 0$, U_3 is an open part of a revolution hypersurface in R^{n+1} around u-axis with a profile curve $c(s)$ in the plane generated by $\{u, v\}$.

First, we prove the following lemmas.

LEMMA 2.1. *Let M^n be a cylindrical hypersurface in R^{n+1} over a unit speed plane curve $c(s) = (c_1(s), c_2(s))$. Then M satisfies (1.5) with $\lambda = 0$, $b = (b_1, \cdots, b_{n+1})$ and $u = (0, 1, 0, \cdots, 0)$ if and only if $b_3 = \cdots = s_{n+1} = 0$ and $c_1''(s) = -b_1$. In particular, for $b_1 = 0$, $c(s)$ is a straight line and for $b_1 \neq 0$, $c(s)$ is defined on the domain (s_0, s_1) such that the curvature $k(s)$ tends to $\pm\infty$ as s tends to s_0, s_1.*

Proof. For the coordinate x of M defined by x (s, t_2, $\cdots$, t_n) = ($c_1(s)$, $c_2(s)$, t_2, $\cdots$, t_n), we have

$$\Delta x = -(c_1''(s), c_2''(s), 0, \cdots, 0) = (b_1, \cdots, b_{n+1}) + g(0, 1, 0, \cdots, 0).$$

Hence the first part follows. Note that for $b_1 \neq 0$,

$$c_1'(s) = -b_1(s-d), \quad c_2'(s) = \pm\sqrt{1 - b_1^2(s-d)^2}, \quad \frac{-1}{|b_1|} < s - d < \frac{1}{|b_1|}.$$

Thus we obtain

$$\kappa(s) = c_1'(s)c_2''(s) - c_1''(s)c_2'(s) = \pm b_1 / \sqrt{1 - b_1^2(s-d)^2}.$$

From this the last part follows.

LEMMA 2.2. *Let M^n be a revolution hypersurface in R^{n+1} around x_{n+1} - axis with a unit speed profile curve $c(s) = (r(s), 0, \cdots, 0, z(s))$, $r(s) > 0$. Then M satisfies (1.5) with $\lambda \neq 0$, $b = 0$ and $u = (0, \cdots, 0, 1)$ if and only if*

$$n\alpha z' = \lambda r, \text{ where } n\alpha = r'z'' - r''z' + (n-1)z'/r, \tag{2.9}$$

or equivalently, for $y = r^n$

$$y'' + n\lambda y = n(n-1)y^{\frac{n-2}{n}}. \tag{2.10}$$

Proof. Let x be the coordinate of M given by $x(s, \theta_2, \cdots, \theta_n) = (r\sin\theta_n \cdots \sin\theta_3 \sin\theta_2, r\sin\theta_n \cdots \sin\theta_3 \cos\theta_2, \cdots, r\cos\theta_n, z)$. We put

$$G = (-z'\sin\theta_n \cdots \sin\theta_3 \sin\theta_2, -z'\sin\theta_n \cdots \sin\theta_3 \cos\theta_2, \cdots, -z'\cos\theta_n, r'),$$

then G is a unit normal vector field of M. Note that

$$\mu_1 = r'z'' - r''z', \quad \mu_2 = \cdots = \mu_n = z'/r.$$

Hence we have $n\alpha = r'z'' - r''z' + (n-1)z'/r$ and

$$\begin{aligned}\Delta x &= -n\alpha G \\ &= n\alpha(z'\sin\theta_n \cdots \sin\theta_3 \sin\theta_2, z'\sin\theta_n \cdots \sin\theta_3 \cos\theta_2, \cdots, z'\cos\theta_n, -r'),\end{aligned}$$

from which the lemma follows.

3. Main Theorems

Now we prove the main theorems.

THEOREM 3.1. *Let M^n be a complete hypersurface of R^{n+1}. Then M^n satisfies (1.5) with $\lambda = 0$ if and only if M^n is minimal in R^{n+1}.*

Proof. Note that in this case, we have $U_2 = U_3$. Suppose that $U_2 = \phi$. Then α is a constant on U_1. And (2.3), (2.4) and (2.5) show that $\mu_1 = \cdots = \mu_n = 0$. Hence we see that $U_1 = \phi$ and M is a minimal hypersurface in R^{n+1}. Now suppose that $U_2 \neq \phi$. Then U_2 is an open part of a cylindrical hypersurface in R^{n+1} over a curve $c(s)$ in the plane parallel with u. Thus Lemma 2.1 and the hypothesis show that M is a hyperplane in R^{n+1}. This contradiction completes the proof of the only if part.

The converse is obvious.

THEOREM 3.2. *Let M be a complete surface of R^3. Then M satisfies (1.5) with $\lambda \neq 0$ if and only if M is, up to congruences, one of the following surfaces:*

1) a sphere $S^2(r)$,

2) a circular cylinder $S^1(r) \times R$,

3) a minimal surface in R^3,

4) a revolution surface around x_3-axis of a unit speed profile curve $c(s) = (r(s), 0, z(s))$, where $r(s)$ is given by

$$(3.1) \qquad r^2 = a\cos\sqrt{2\lambda}s + \frac{1}{\lambda}, \quad 0 < a < \frac{1}{\lambda}.$$

Proof. Suppose that $U_2 = \phi$, then α is a constant on U_1. Hence if $U_1 = \phi$,Takahashi's theorem implies that $M$is either minimal or a sphere in R^3. And if $U_1 \neq \phi$, it can be shown that U_1 is an open part of a circular cylinder. Since such surfaces cannot be pasted to be a globally smooth one, we see that, in this case, $M = U_1$ is a circular cylinder.

Now suppose that $U_3 \neq \phi$. Then we may assume that U_3 is an open part of a revolution surface with z-axis of a unit speed profile curve $c(s) = (r(s), 0, z(s))$. And Lemma 2.2 shows that $c(s)$ satisfies the following:

$$(3.2) \qquad y'' + 2\lambda y = 2, y = r^2, \quad r > 0 \text{ and } (r')^2 \leq 1.$$

For $\lambda > 0$, by solving the linear equation (3.2), we see that the following, up to reparametrizations of s, are the only solutions of (3.2)

$$(3.3) \qquad y_1 = \frac{1}{\lambda}, \quad y_2 = \frac{2}{\lambda}\cos^2(\sqrt{\frac{\lambda}{2}}s), \quad y_3 = a\cos\sqrt{2\lambda}s + \frac{1}{\lambda}.$$

Note that y_1 and y_2 represent a circular cylinder and a sphere, respectively, and in these cases,we have $U_2 = \phi$. And note that as $a \to 0$ and $\frac{1}{\lambda}$, $y_3 \to y_1$ and y_2, respectively.

For $\lambda < 0$, every solution $r(s)$ of (3.2) has a value $s = s_0$ such that $|r'(s_0)| = 1$ and $r''(s_0) \neq 0$. Since the curvature $\kappa(s)$ of $c(s)$ is given by

$$\kappa(s) = r'(s)z''(s) - r''(s)z'(s) = \pm\frac{r''(s)}{\sqrt{1 - r'(s)^2}},$$

we see that $\kappa(s) \to \pm\infty$ as $s \to s_0$. Thus the solution $r(s)$ cannot be extended to be a global solution of (3.2).

Note that open parts of the surfaces in the theorem cannot be pasted to be a smooth complete one. This fact implies that U_3 is the entire surface M. Therefore M is the revolution surface given in 4). The converse is obvious.

Since minimal surfaces are not compact, above theorems imply the following:

COROLLARY 3.3. *The spheres are the only compact surfaces in R^3 satisfying (1.5).*

References

1. B. Y. Chen, *Total mean curvature and submanifolds of finite type*, World Scientific, New Jersey and Singapore (1984).
2. B. Y. Chen and M. Petrovic, *On spectral decomposition of immersions of finite type*, Bull. Austral. Math. Soc. **44** (1991), 117-129.
3. O. J. Garay, *An extension of Takahashi's theorem*, Geometriae Dedicata **34** (1990), 105-112.
4. TH. Hasanis and TH. Vlachos, *Hypersurfaces of E^{n+1} satisfying $\Delta x = Ax + B$*, J. Austral. Math. Soc. (Series A) **53** (1992), 377-384.
5. T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan. **18** (1966), 380-385.