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ON ENDOMORPHISMS OF MULTIPLICATION MODULES

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1. Introduction

Let R be a commutative ring with identity and M a unital R-module. The module M is said to be a *multiplication module* provided for each submodule N of M there exists an ideal I of R such that N=IM. It is clear that every cyclic R-module is a multiplication module. Let End(M) be the ring of R-homomorphisms of M. An endomorphism φ of M will be called *trivial* if there exists $a \in R$ such that $\varphi(m) = am$ for all $m \in M$. W. Vasconcelos[8] proved that for a commutative ring R, injective endomorphisms of finitely generated R-modules are isomorphisms if and only if every prime ideal of R is maximal. Also, J. Strooker[7] and Vasconcelos[8] independently proved that for a commutative ring R, surjective endomorphisms of finitely generated R-modules are isomorphisms. In this paper, We relate these concepts to multiplication module terminology. Also we will show that if M is a finitely generated multiplication module, then every endomorphism of M is trivial (See Theorem 3) and some other related properties of endomorphisms of multiplication modules are studied.

2. Preliminaries

Let R be a commutative ring with identity. Let M be an R-module. A submodule K of M is called *fully invariant* if $\varphi(K) \subseteq K$, for every endomorphism φ of M. For any submodule N of M, We set $(N : M) = \{r \in R : rM \subseteq N\}$. Note that (N:M) is the annihilator of the R-module M/N and is an ideal of R. Let N be a submodule of a multiplication module M. There exists an ideal I of R such that N=IM. Note that

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I$\subseteq$ $(N : M)$ and N=IM$\subseteq$ $(N : M)M \subseteq N$ so that N=(N:M)M. It follows that an R-module M is a multiplication module if and only if N=(N:M)M for all submodules N of M.

3. Endomorphisms of multiplication modules

Our starting point is the following result taken from [6,Proposition 7].

LEMMA 1. *Let M be a multiplication module. Then*
 (i) *Every submodule of M is fully invariant.*
 (ii) $\varphi \in End(M)$ *is an epimorphism if and only if* $(\varphi|N) : N \to N$ *is an epimorphism for all submodules N of M.*

Note that Lemma 1 (ii) gives at once that every epimorphism of a multiplication module is an automorphism.

LEMMA 2. *Let* $M_i(1 \leq i \leq n)$ *be a finite collection of submodules of a module M such that* $M = M_1 + \cdots + M_n$. *Let* φ *be an endomorphism of M such that* $\varphi(M_i) \subseteq M_i(1 \leq i \leq n)$. *Let* r$\in$ $(M_1 : M) + \cdots + (M_n : M)$. *Suppose that the restriction of* φ *to* M_i *is trivial for each* $1 \leq i \leq n$. *Then the endomorphism* $r\varphi$ *of M is trivial.*

Proof. For each $1 \leq i \leq n$, there exist $a_i \in R$ such that $\varphi(x) = a_i x$ for each $x \in M_i$. There exists $r_i \in (M_i : M)(1 \leq i \leq n)$ such that $r = r_1 + \cdots + r_n$. Let $m \in M$. Then

$$\begin{aligned} r\varphi(m) &= r_1\varphi(m) + \cdots + r_n\varphi(m) \\ &= \varphi(r_1 m) + \cdots + \varphi(r_n m) \\ &= a_1 r_1 m + \cdots + a_n r_n m \\ &= (a_1 r_1 + \cdots + a_n r_n)m \end{aligned}$$

, for all $m \in M$. Thus $r\varphi$ is trivial.

THEOREM 3. *Let M be a finitely generated multiplication module. Then every endomorphism of M is trivial.*

Proof. By [1,Theorem 1], there exist a positive integer n and elements $m_i \in M$ such that $R = (Rm_1 : M) + \cdots + (Rm_n : M)$. Let

φ be any endomorphism of M. For any $1 \leq i \leq n$, Lemma 1 gives $\varphi(Rm_i) \subseteq Rm_i$ and hence φ is trivial on Rm_i. By Lemma 2, $\varphi = 1\varphi$ is trivial on M.

The following example shows that if finitely generated assumption in Theorem 3 is deleted then it is not true in general.

EXAMPLE 4. Let K be a field, R the direct product of a countably infinite number of copies of K, so R=$\{(a_1, a_2, a_3, \cdots) : a_i \in K(i \geq 1)\}$. Let I=$\oplus K$, an ideal of R generated by idempotents. Thus I is a multiplication ideal. Define $\varphi : I \to I$ by $\varphi(a_1, a_2, a_3, \cdots) = (0, a_1, a_2, a_3, \cdots)$ where $a_i \in K(i \geq 1), a_i \neq 0$ for only finitely many i. Then there does not exist $r \in R$ such that $\varphi(x) = rx(x \in I)$. However , for any finitely generated subideal J of I every endomorphism $\theta : J \to J$ is trivial.

PROPOSITION 5. *Let M be a finitely generated module. Then the following statements are equivalent.*

(i) *M is a multiplication module.*
(ii) *Every endomorphism of every homomorphic image of M is trivial.*
(iii) *Every submodule of every homomorphic image of M is fully invariant.*

Proof. (i)$\Rightarrow$(ii) By Theorem 3, Since every homomorphic image of M is a finitely generated multiplication module.

(ii) $\Rightarrow$(iii) Trivial.

(iii) $\Rightarrow$(i) Let P be any maximal ideal of R. Then M/PM is a vector space over the field R/P and hence M/PM is semisimple. Because every submodule of M/PM is fully invariant, it follows that M/PM is cyclic. By [5, Corollary 1.5], M is a multiplication module.

DEFINITION 6. A module M is said to satisfy *Fitting's Lemma* if for each $\varphi \in End(M)$ there exists an integer $n \geq 1$ such that $M = Ker\varphi^n \oplus Im\varphi^n$.

DEFINITION 7. A ring R is left(right) *π-regular* if for each a in R there exists b in R and an integer $n \geq 1$ such that $a^n = ba^{n+1} (a^n = a^{n+1}b)$.

THEOREM 8[6]. *Let M be a multiplication module satisfying descending chain condition on multiplication submodules and $\varphi \in End$ (M). Then M satisfies Fitting's Lemma.*

Proof. Consider the sequence $M \supset \varphi(M) \supset \varphi^2(M) \cdots$. Since every homomorphic images of multiplication modules are multiplication ones, the sequence becomes stationary after n steps, say. Thus $\varphi^n(M) = \varphi^{n+1}(M) = \cdots = \varphi^{2n}(M) = \cdots$. Therefore φ^n induces an endomorphism on multiplication module $\varphi^n(M)$ which is an epimorphism, hence an automorphism because every epimorphism of a multiplication module onto itself is an automorphism. Thus $\varphi^n(M) \cap Ker\varphi^n(M) = 0$. Now take any $m \in M$, then $\varphi^n(m) = \varphi^{2n}(n)$ for some $n \in M$, hence $\varphi^n(m - \varphi^n(n)) = 0$. This means $m - \varphi^n(n) \in Ker(\varphi^n)$. Since $m = \varphi^{(n)}(n) + (m - \varphi^n(n))$, $M = \varphi^{(n)}(M) \oplus Ker(\varphi^n)$. The theorem is proved.

Following Azumaya[4], a ring which is both left and right π-regular will be called *strongly π-regular.* However, F. Dischinger proved that every left π-regular rings are right π-regular and hence strongly π-regular.

COROLLARY 9. *If a multiplication module M satisfies descending chain condition on multiplication submodules, then injective endomorphisms of M are isomorphisms.*

Proof. By Theorem 8 and [3, Proposition 2.3].

Recall that an R-module M is said to be *indecomposable* if it is not the direct sum of two nonzero submodules.

PROPOSITION 10. *Let R be an integral domain and M a torsion-free multiplication module. Then M is indecomposable*

Proof. Suppose that $M_1 \neq 0, M_2 \neq 0$ and $M = M_1 \oplus M_2$. Since M is a multiplication module, so are M_1 and M_2. Also, there exists an ideal I_1 of R such that $I_1M_1 = M_1$ and $I_1M_2 = 0$ (See [5, Theorem 2.2]). This implies $I_1 \subseteq ann(M_2)$. But M is a torsion-free R-module. Therefore we have ann(M_2)=0. Thus $I_1 = 0$ and hence $M_1 = 0$. This is a contradiction. Therefore the result.

THEOREM 11. *Let M be an indecomposable multiplication module satisfying descending chain condition on multiplication submodules and let $\varphi \in End(M)$. Then the followings are equivalent.*

(i) *φ is a monomorphism.*
(ii) *φ is an epimorphism.*
(iii) *φ is an automorphism*
(iv) *φ is not nilpotent.*

Proof. (i) $\Rightarrow$ (ii) Suppose φ is a monomorphism and consider the chain of R-submodules $M \supseteq \varphi(M) \supseteq \varphi^2(M) \supseteq \cdots$. Since M is a multiplication module, so is every homomorphic images of M. By hypothesis, this chain will terminate after a finite number of steps, say n steps, then $\varphi^n(M) = \varphi^{n+1}(M)$. Given an arbitrary $x \in M$, $\varphi^n(x) = \varphi^{n+1}(y)$ for some $y \in M$. As φ is assumed to be a one-to-one function, φ^n also enjoys this property, whence $x = \varphi(y)$. This implication is that $M = \varphi(M)$ and so φ is an epimorphism.

(ii) $\Rightarrow$ (i) Suppose φ is an epimorphism. Since $Ker\varphi$ is a submodule of M, $Ker\varphi = IM$ for some ideal I of R. Thus $0 = \varphi(Ker\varphi) = \varphi(IM) = I\varphi(M) = IM = Ker\varphi$ and hence φ is a monomorphism.

(ii) $\Leftrightarrow$ (iii) By Lemma 1 (ii).

(iii) $\Rightarrow$ (iv) Trivial.

(iv) $\Rightarrow$ (iii) Suppose that φ is not nilpotent and let φ be any endomorphism of M. By Theorem 8 , there exists $n \in Z_+$ such that $M = \varphi^n(M) \oplus Ker(\varphi^n)$. Since M is indecomposable, either Ker(φ^n)=0 or $\varphi^n(M) = 0$. If (φ^n)(M)=0 (i.e. Ker(φ^n)=M for some $n \in Z_+$), then for all $x \in M, \varphi^n(x)$=0. Thus φ is a nilpotent. This is a contradiction and hence Ker(φ^n) =0. This implies φ^n is a monomorphism and so φ is a monomorphism. Consider the sequence $M \supseteq \varphi(M) \supseteq \varphi^2(M) \supseteq \cdots$ .Since Ker(φ^n)=0,$M = \varphi^n(M)$. This shows that $M = \varphi(M)$. Therefore φ is an automorphism.

COROLLARY 12. *Let M be an indecomposable multiplication module satisfying descending chain condition on multiplication submodules and $g = \varphi_1 + \varphi_2 + \cdots + \varphi_n$ is an automorphism, $\varphi_i \in End(M)$. Then for some φ_i is an automorphism.*

Proof. First consider the case $n = 2$. Then $g = \varphi_1 + \varphi_2$. This implies $1 = g^{-1}\varphi_1 + g^{-1}\varphi_2$. By the proof of Theorem 11, $g^{-1}\varphi_1$ is an automorphism or nilpotent. In the second case it can easily be checked that $g^{-1}\varphi_2 = 1 - g^{-1}\varphi_1$ is an automorphism. We now treat the general situation by induction. If φ_1 is not an automorphism, then $g^{-1}\varphi_1$ is not. Since $g^{-1}\varphi_1 \in End(M)$, $g^{-1}\varphi_1$ is nilpotent by Theorem 11. Thus $1 - g^{-1}\varphi_1$ is an automorphism. By inductional assumption, $g^{-1}\varphi_i$ is an automorphism for some $i \neq 1$ and so φ_i is an automorphism for some $i \neq 1$.

PROPOSITION 13. *Let M be an indecomposable multiplication module satisfying descending chain condition on multiplication submodules. Then $End(M)$ is a local ring.*

Proof. Suppose M satisfies the conditions. Let $f, g \in End(M)$. It is sufficient to show that if $f+g$ is invertible in End(M), then f or g is an invertible (See [2], p.170 Proposition 15.15). Suppose $f+g$ is invertible and g is not invertible. Then there exists an automorphism h such that $(f+g)h = 1_M$ in $End(M)$. Since g is not invertible, so is gh. By the proof of Theorem 11, gh is nilpotent. Thus there exists $n \in Z_+$ such that $(gh)^n = 0$. This implies that $(1-gh)(1+gh+\cdots+(gh)^{n-1}) = 1$ and hence fh is invertible. Accordingly f is invertible.

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