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SOME ANALYTIC CLASSIFICATION OF PLANE CURVE SINGULARITIES TOPOLOGICALLY EQUIVALENT TO THE EQUATION $z^n + y^k = 0$ WITH $gcd(n,k) = 1$

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1. Introduction

We know [6] that the analytic classification of complex hypersurfaces with isolated singularity at the origin is the same as the algebraic classification of their corresponding moduli algebra over the complex field. In fact, even an algebraic classification of irreducible plane curve singularities at the origin is a very delicated and complicated problem. For example, consider the family of analytic irreducible plane curve singularities f_α at the origin parametrically defined by $y = t^4$ and $z = t^9 + t^{10} + \alpha t^{11}$ where α is a number. Then for any α f_α is clearly topologically equivalent to the equation $z^4 + y^9 = 0$ at the origin, but for any two numbers $\alpha \neq \beta$ f_α and f_β are analytically different at the origin [2].

Let $V = \{f(z,y) = 0\}$ and $W = \{z^n + y^k = 0\}$ with $gcd(n,k) = 1$ be analytic irreducible plane curves with isolated singularities at the origin. Assume that V and W are topologically equivalent at the origin. Then denote this relation by $f \sim z^n + y^k$ for brevity. So by a nonsingular linear change of coordinates f can be written as $u(z^n + a_2 y^{\alpha_2} z^{n-2} + \cdots + a_{n-1} y^{\alpha_{n-1}} z + y^k)$ where u is a unit and the $a_i = a_i(y)$ are units in ${}_2\mathcal{O}$, the ring of germs of holomorphic functions at the origin in $\mathbb{C}^2$ and $\dfrac{\alpha_i}{i} > \dfrac{k}{n}$ for $i = 2, \ldots, n-1$ [5]. If V and W are analytically equivalent at the origin, then denote this relation by $f \approx g$. If not, we write $f \not\approx g$.

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Then we are going to prove the following cases:

(1) If $f = z^n + uy^\alpha z^\beta + y^k \sim z^n + y^k$ where u is a unit in ${}_2\mathcal{O}$, then $f \approx z^n + y^\alpha z^\beta + y^k$.

(2) If $f = z^n + uy^\alpha z^\beta + y^k$, $g = z^n + vy^\gamma z^\delta + y^k$ and $f \sim g \sim z^n + y^k$ where u, v are units in ${}_2\mathcal{O}$ and $1 \leq \alpha,\ \gamma \leq k-2$ and $1 \leq \beta$, $\delta \leq n-2$, then $f \approx g$ if and only if $a = \gamma$ and $\beta = \delta$.

(3) In the case (2), if $1 \leq \alpha \leq k-1$ and $1 \leq \beta \leq n-1$, then $f \approx g$ does not imply that $\alpha = \gamma$ or $\beta = \delta$.

(4) Suppose that $f \sim z^n + y^k$ and $g \sim z^n + y^k$. By [1], $f \approx z^n + y^k + \sum c_i P_i$ and $g \approx z^n + y^k + \Sigma d_j Q_j$ where each c_i and d_j are nonzero numbers if exist and $P_i = y^{\alpha_i} z^{\beta_i}$, $Q_j = y^{\gamma_j} z^{\delta_j}$; $1 \leq \alpha_i, \gamma_j \leq k-2$; $1 \leq \beta_i, \delta_j \leq n-2$ satisfying that $n\alpha_i + k\beta_i > nk$ and $n\gamma_j + k\delta_j > nk$. Let $m(f) = \mathrm{Min}\{\alpha_i + \beta_i ; c_i \neq 0\}$ and $m(g) = \mathrm{Min}\{\gamma_j + \delta_j : d_j \neq 0\}$. If $f \approx g$, then $\{(\alpha_i, \beta_i) : \alpha_i + \beta_i = m(f)\} = \{(\gamma_j, \delta_j) : \gamma_j + \delta_j = m(g)\}$ as sets.

(5) Let $f = z^n + y^k + \Sigma c_i P_i$ where each c_i is a nonzero number if exists and $P_i = y^{\alpha_i} z^{\beta_i}$ with $n\alpha_i + k\beta_i > nk$ and $1 \leq \alpha_i \leq k-2$, $1 \leq \beta_i \leq n-2$. Then $f \approx z^n + y^k$ if and only if all c_i are zero.

2. Known preliminaries

DEFINITION 2.1. Let $V = \{z \in \mathbb{C}^n : f(z) = o\}$ and $W = \{z \in \mathbb{C}^n : g(z) = o\}$ be germs of complex analytic hypersurfaces with isolated singular points at the origin. (i) V and W are said to be topologically equivalent at the origin if there is a germ at the origin of homeomorphisms $\phi : (U_1, 0) \to (U_2, 0)$ such that $\phi(V) = W$ and $\phi(0) = 0$ where U_1 and U_2 are open subset containing the origin in $\mathbb{C}^n$. In this case denote this relation by $f \sim g$. (ii) V and W are said to be analytically equivalent at the origin if there is a germ at the origin of biholomorphisms $\psi : (U_1, 0) \to (U_2, 0)$ such that $\psi(V) = W$ and $\psi(0) = 0$ where U_1 and U_2 are open subsets containing the origin in $\mathbb{C}^n$. Then denote this relation by $f \approx g$. If not, we write $f \not\approx g$. Let ${}_n\mathcal{O}$ denote the ring of germs of holomorphic functions at the origin in $\mathbb{C}^n$.

THEOREM 2.2 [5]. *Let* $f(z,y) = a_0 z^n + a_1 y^{\alpha_1} z^{n-1} + \cdots + a_n y^{\alpha_n}$ *be irreducible in* ${}_2\mathcal{O}$ *where each* a_i *is a unit in* ${}_2\mathcal{O}$ *if exists and the* α_i

are positive integers. Then $\frac{\alpha_i}{i} \geq \frac{\alpha_n}{n}$ *for all* i. *Moreover, if* $\alpha_n = nk$ *for some integer* k, *then* $\frac{\alpha_n}{n} = \frac{\alpha_i}{i}$ *for all* $i = 1, \ldots, n-1$.

COROLLARY 2.3. *Let* $f(z,y) = z^n + a_1 y^{\alpha_1} z^{n-1} + \cdots + a_{n-1} y^{\alpha_{n-1}} z + y^k$ *with* $(n,k) = 1$ *where* $a_i = a_i(y)$ *is a unit in* ${}_2\mathcal{O}$ *if exists and the* α_i *are positive integers. Then* f *is irreducible in* ${}_2\mathcal{O}$ *if and only if* $\frac{k}{n} < \frac{\alpha_i}{i}$ *for all* $i \neq n$. *Moreover, in this case* $f \sim z^n + y^k$ *in* ${}_2\mathcal{O}$.

DEFINITION 2.4. The polynomial $f(z_1, \ldots, z_n)$ is called weighted homogeneous of type $(\frac{1}{a_1}, \ldots, \frac{1}{a_n})$ if there is a some positive rational numbers $a_1, \ldots, a_n$ such that $f(t^{a_1} z_1, \ldots, t^{a_n} z_n) = t f(z_1, \ldots, z_n)$.

THEOREM 2.5 (MATHER-YAU [6]). *Suppose that* $V = \{f(z_1, \ldots, z_n) = 0\}$ *and* $W = \{g(z_1, \ldots, z_n) = 0\}$ *have the isolated singular point at the origin. Then the following conditions are equivalent:*

(i) $f \approx g$.

(ii) $A(f)$ *is isomorphic to* $A(g)$ *as a* $\mathbb{C}$*-algebra where* $A(f) = {}_n\mathcal{O}/(f, \Delta(f))$, $A(g) = {}_n\mathcal{O}/(g, \Delta(g))$ *and* $(f, \Delta(f))$ *is the ideal in* ${}_n\mathcal{O}$ *generated by* $f, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}$.

(iii) $B(f)$ *is isomorphic to* $B(g)$ *as a* $\mathbb{C}$*-algebra where* $B(f) = {}_n\mathcal{O}/(f, m\Delta(f))$, $B(g) = {}_n\mathcal{O}/(g, m\Delta(g))$ *and* $(f, m\Delta(f))$ *is the ideal in* ${}_n\mathcal{O}$ *generated by* f *and* $z_i \frac{\partial f}{\partial z_j}$ *for all* $i, j = 1, \ldots, n$.

THEOREM 2.6 (ARNOLD [1]). *Assume that* $n < k$, $(n,k) = 1$ *and that* $g = z^n + a_1 y^{\alpha_1} z^{n-1} + \cdots + a_{n-1} y^{\alpha_{n-1}} z + y^k \sim z^n + y^k$ *at the origin in* $\mathbb{C}^2$ *where each* $a_i = a_i(y)$ *is a unit in* ${}_2\mathcal{O}$ *if exists and the* α_i *are positive integers. Then* $g \approx z^n + y^k + \Sigma c_i P_i$ *where each* c_i *is a nonzero number if exists and* $P_i = y^{\alpha_i} z^{\beta_i}$ *with* $1 \leq \beta_i \leq n-2$ *and* $1 \leq \alpha_i \leq k-2$ *with* $n\alpha_i + k\beta_i > nk$.

3. Some analytic classification of irreducible plane curve singularities

THEOREM 3.1. *Let $f = z^n + uy^\alpha z^\beta + y^k$ with $n\alpha + k\beta > nk$ where $n < k$, $(n, k) = 1$ and $u = u(z, y)$ is a unit in ${}_2\mathcal{O}$. Then $f \sim z^n + y^k$ and $f \approx z^n + y^\alpha z^\beta + y^k$.*

Proof. By Corollary 2.3, clearly $f \sim z^n + y^k$. First note that for any number $c \neq 0$ $f_c(z, y) = z^n + cy^\alpha z^\beta + y^k \approx z^n + y^\alpha z^\beta + y^k$ because $f_c(t^k z, t^n y) = t^{nk}(z^n + ct^{n\alpha+k\beta-nk}y^\alpha z^\beta + y^k)$.

(i) To show that $f \approx z^n + y^\alpha z^\beta + y^k$, first assume that $u(z, y) = u(z, 0)$ is a unit in ${}_2\mathcal{O}$. Then by a nonsingular linear change of coordinates at the origin, $f = z^n + u(z, 0)y^\alpha z^\beta + y^k \approx v(z)z^n + y^\alpha z^\beta + y^k = h$ where $v = v(z)$ is a unit in ${}_2\mathcal{O}$. Now it is enough to show that $h \approx v(0)z^n + y^\alpha z^\beta + y^k = g$. To use Theroem 2.5 compute the ideal $I = (h, m\Delta(h))$ in ${}_2\mathcal{O}$ generated by $h, zh_z, yh_y, yh_z.zh_y$ as follows:

$$\begin{aligned}
h &= vz^n + y^\alpha z^\beta + y^k \\
zh_z &= (zv' + nv)z^n + \beta y^\alpha z^\beta \\
yh_y &= \alpha y^\alpha z^\beta + ky^k \\
yh_z &= (zv' + nv)yz^{n-1} + \beta y^{\alpha+1} z^{\beta-1} \\
zh_y &= \alpha y^{\alpha-1} z^{\beta+1} + ky^{k-1}z.
\end{aligned}$$

Then solve the equation $h \equiv zh_z \equiv yh_y \equiv 0 \pmod I$ with respect to z^n, $y^\alpha z^\beta$ and y^k as below:

$$\begin{vmatrix} v & 1 & 1 \\ zv' + nv & \beta & 0 \\ 0 & \alpha & k \end{vmatrix} = v(n\alpha + k\beta - nk) + zv'(k - \alpha) \neq 0$$

at the origin. Thus $z^n, y^\alpha z^\beta, y^k$ belong to I. Considering yh_z and zh_y, then $I = (z^n, y^\alpha z^\beta, y^k, nv(0)z^{n-1} + \beta y^{\alpha+1} z^{\beta_1},\ \alpha y^{\alpha-1} z^{\beta+1} + ky^{k-1}z)$. So $I = (g, m\Delta(g))$. By Theorem 2.5 $f \approx v(0)z^n + y^\alpha z^\beta + y^k \approx z^n + y^\alpha z^\beta + y^k$.

(ii) Now let $f = z^n + u(z, y)y^\alpha z^\beta + y^k$. Also by a nonsingular linear change of coordinates, it is enough to consider $\ell(z, y) = z^n + y^\alpha z^\beta +$

$v(z,y)y^k$ where $v = v(z,y)$ is a unit in ${}_2\mathcal{O}$. Then compute the ideal $J = (\ell, m\Delta(\ell))$ as follows:

$$\begin{aligned}
\ell &= z^n + y^\alpha z^\beta + vy^k \\
z\ell_z &= nz^n + \beta y^\alpha z^\beta + v_z y^k z \\
y\ell_y &= \alpha y^\alpha z^\beta + (yv_y + kv)y^k \\
y\ell_z &= nyz^{n-1} + \beta y^{\alpha+1} z^{\beta-1} + v_z y^{k+1} \\
z\ell_y &= \alpha y^{\alpha-1} z^{\beta+1} + (yv_y + kv)y^{k-1} z.
\end{aligned}$$

Similarly as the previous case, solve the equation $\ell \equiv z\ell_z \equiv y\ell_y \equiv 0 \pmod{\mathrm{J}}$ relative to $z^n, y^\alpha z^\beta$ and y^k. Then we prove easily that $z^n, y^\alpha z^\beta$ and y^k belong to J. Considering $y\ell_z$ and $z\ell_y$, then $J = (z^n, y^\alpha z^\beta, y^k, nyz^{n-1} + \beta y^{\alpha+1} z^{\beta-1}, \alpha y^{\alpha-1} z^{\beta+1} + kv(z,0)y^{k-1}z)$ where $v(z,0)$ is a unit in ${}_2\mathcal{O}$. Therefore $\ell \approx z^n + y^\alpha z^\beta + v(z,0)y^k$ by Theorem 2.5. By another nonsingular linear change of coordinates at the origin, $z^n + y^\alpha z^\beta + v(z,0)y^k \approx z^n + w(z)y^\alpha z^\beta + y^k$ where $w(z)$ is a unit in ${}_2\mathcal{O}$. By (i), we proved the theorem.

THEOREM 3.2. *Let $f(z,y) = z^n + y^k + uy^\alpha z^\beta$ and $g = z^n + y^k + vy^\gamma z^\delta$ where $n < k$, $(n,k) = 1$ and $u = u(z,y)$, $v = v(z,y)$ are units in ${}_2\mathcal{O}$ and $1 \le \beta$, $\delta \le n-2; 1 \le \alpha$, $\gamma \le k-2$ with $n\alpha + k\beta > nk$ and $n\gamma + k\delta > nk$. Then $f \approx g$ if and only if $\alpha = \gamma$ and $\beta = \delta$.*

Proof. By Theroem 3.1, if $\alpha = \gamma$ and $\beta = \delta$ then $f \approx g$. Now suppose that $f \approx g$. Then by Theorem 3.1, we may put $f = z^n + y^k + y^\alpha z^\beta$ and $g = z^n + y^k + y^\gamma z^\delta$. Let us prove the condition that $\alpha = \gamma$ and $\beta = \delta$. Assume the contrary. So it is enough to consider the following cases : (I) $\alpha + \beta < \gamma + \delta$ and (II) $\alpha + \beta = \gamma + \delta$ with $\alpha \neq \gamma$.

By definition, if $f \approx g$ then there is a biholomorphic mapping $\phi : (U_1, 0) \to (U_2, 0)$ such that $uf = g \circ \phi$, where U_1 and U_2 are open subsets containing the origin and u is a unit in ${}_2\mathcal{O}$. Write $\phi(z,y) = (H, L)$ as follows:

$$\begin{aligned}
H &= H(z,y) = az + by + H_2 + H_3 + \cdots \quad \text{and} \\
L &= L(z,y) = cz + dy + L_2 + L_3 + \cdots
\end{aligned}$$

where H_n and L_n are homogeneous polynomials of degree n with $H_n = H_n(z,y) = a_{n,0}z^n + a_{n-1,1}z^{n-1}y + \cdots + a_{0,n}y^n$ and $L_n = L_n(z,y) = b_{n,0}z^n + b_{n-1,1}z^{n-1}y + \cdots + b_{0,n}y^n$.

Note that $ad - bc \neq 0$. Then $g \circ \phi(z,y) = (az + by + H_2 + H_3 + \cdots)^n + (cz + dy + L_2 + L_3 + \cdots)^\gamma(az + by + H_2 + H_3 + \cdots)^\delta + (cz + dy + L_2 + L_3 + \cdots)^k = u(z^n + y^\alpha z^\beta + y^k)$ where u is a unit in ${}_2\mathcal{O}$. We know that $b = 0$ because $n < k$, $\alpha + \beta > n$ and $\gamma + \delta > n$. Let us prove the first case (I).

(I) We are going to separate this fact into the following three cases : (i) $\alpha + \beta < k$, (ii) $\alpha + \beta = k$, (iii) $\alpha + \beta > k$.

(i) $\alpha + \beta < k$: Observe that $H_2, H_3, \cdots, H_{\alpha+\beta-n}$ can be analytically divisible by z in the expansion of H^n in $g \circ \phi(z,y)$, considering ${}_2\mathcal{O}$ as a unique factorization ring up to a unit [3]. Therefore in the expansion of $H^n = (az{\cdot}unit + H_{\alpha+\beta-n+1} + \cdots)^n$, we cannot find a nonzero monomial $y^\alpha z^\beta$ with $\beta \leq n-2$ because $\beta + (n-\beta)(\alpha+\beta-n+1) > \alpha+\beta$ if and only if $(\alpha+\beta-n)(n-\beta-1) > 0$.

(ii) $\alpha + \beta = k$: Observe also that $H_2, H_3, \cdots, H_{k-n}$ can be analytically divisible by z in the expansion of H^n in $g \circ \phi(z,y)$. To find the set of nonzero monomials $y^\ell z^m$ with $\ell + m = k$ and $m \leq n-1$ in the expansion of $g \circ \phi(z,y)$, it is enough to consider the following:

$$n(az)^{n-1}H_{k-n+1} + (cz + dy)^k.$$

Since the coefficient of monomial $y^{k-1}z$ must be zero, $ad - bc \neq 0$ implies that $c = 0$. Therefore we cannot find a nonzero monomial $y^\alpha z^\beta$ in the expansion of $g \circ \phi(z,y)$ because $\beta \leq n-2$.

(iii) $\alpha + \beta > k$: Note that $H_2, H_3, \cdots, H_{k-n}$ can be analytically divisible by z in the expansion of H^n in $g \circ \phi(z,y)$. Since there is no nonzero monomial $y^{k-1}z$ in uf, $\alpha + \beta > k$ implies that $c = 0$ and also H_{k-n+1} can be divided by z analytically in ${}_2\mathcal{O}$. So in the expansion of $g \circ \phi(z,y)$ to find a nonzero monomial $y^\alpha z^\beta$, it is sufficient to consider the following:

$$(az \cdot unit + H_{k-n+2} + H_{k-n+3} + \cdots)^n + (dy + L_2 + L_3 + \cdots)^k.$$

Now we are going to show that (iii$_\mathrm{a}$)there is no nonzero term $y^\alpha z^\beta$ in the expansion of H^n and (iii$_\mathrm{b}$) there is no nonzero term $y^\alpha z^\beta$ in the expansion of L^k. Consider the case (iii$_\mathrm{a}$). If H_j can be analytically divisible by z for all j, there is nothing to prove. If not, let m be the smallest positive integer such that H_m cannot be analytically divisible by z. Then in the expansion of H^n, $z^{n-1}H_m$ contains a nonzero monomial $z^{n-1}y^m$. If $n-1+m \geq \alpha+\beta$, there is no nonzero term $y^\alpha z^\beta$ in the expansion of H^n. If $n-1+m < \alpha+\beta$ and there is no nonzero monomial $y^m z^{n-1}$ in the expansion of L^k, then we cannot find a nonzero term $y^m z^{n-1}$ in uf, because if exists then $\alpha \leq m$ and $\beta \leq n-2$ would imply that $\alpha+\beta \leq m+n-2 < \alpha+\beta$. It is a contradiction. So it is enough to prove that if $n-1+m < \alpha+\beta$ and there is a nonzero term $y^m z^{n-1}$ in L^k, we could find a contradiction. Then let r be the smallest positive integer such that L_r cannot be analytically divisible by y. Thus we would get some inequality as follows:

$$n-1+m \geq m+r(k-m).$$

Claim that $n-1+m = m+r(k-m)$. If $n-1+m > m+r(k-m)$, then there is neither a nonzero term $y^m z^{r(k-m)}$ in uf nor in H^n because $k+n-4 \geq \alpha+\beta > n-1+m > m+r(k-m)$ implies that $k-3 > m$ and $r(k-m) < n-1$. Since there exists a nonzero term $y^m z^{r(k-m)}$ in $g \circ \phi(z,y)$, it would be a contradiction. Therefore we get the equation: (A) $n-1+m = m+r(k-m)$.

Now consider a nonzero term $y^{k-1}z^r$ in L^k. Note that $k-1+r < \alpha+\beta$ because $\alpha+\beta > n-1+m = m+r(k-m)$ and $m+r(k-m)-(k-1+r) = (k-m-1)(r-1) > 0$. Since there is no nonzero monomial $y^{k-1}z^r$ in uf, if there is no nonzero term $y^{k-1}z^r$ in H^n then there is nothing to prove. If there is a nonzero term $y^{k-1}z^r$ in H^n, by the similar method just as before, we get another equation: (B) $k-1+r = r+m(n-r)$. From two equations (A) and (B), $k(1+r) = n(m+1)$, i.e., $\dfrac{k}{n} = \dfrac{m+1}{r+1}$. Note that $(n,k) = 1$. So the equation $k(1+r) = n(m+1)$ does not hold because $k-1+r < \alpha+\beta \leq n+k-4$ implies $r < n-3$. Thus we proved the case (iii$_\mathrm{a}$).

Next, using the similar technique as in the case (iii$_\mathrm{a}$), we can prove the case (iii$_\mathrm{b}$). Thus we proved the theorem in the case (I).

Similarly, we can prove the case (II).

In the assumption of Theorem 3.2, if $f = z^n + y^k + y^\alpha z^\beta \sim z^n + y^k$ with $1 \leq \alpha \leq k-1$ and $1 \leq \beta \leq n-1$, then we can prove that the result of Theorem 3.2 may not be true by the following examples:

(1) Let $f = z^4 + y^8 z + y^9$ and $g = z^4 + y^7 z^2 + y^9$. Then $f \approx g$ but $f \not\approx z^4 + y^9$ by Theorem 2.5.

(2) Let $f = z^4 + y^4 z^3 + y^9$ and $g = z^4 + y^8 z^2 + y^9$. Then $f \approx g \approx z^4 + y^9$ by Theorem 2.5 and Theorem 3.1.

THEOREM 3.3. *Let $f = z^n + y^k + \Sigma c_i P_i$ and $g = z^n + y^k + \Sigma d_j Q_j$ where $n < k$, $(n,k) = 1$ and each c_i and d_j are nonzero numbers if exist and $P_i = y^{\alpha_i} z^{\beta_i}$, $Q_j = y^{\gamma_j} z^{\delta_j}$ with $1 \leq \alpha_i, \gamma_j \leq k-2$ and $1 \leq \beta_i$, $\delta_j \leq n-2$ satisfying that $n\alpha_i + k\beta_i > nk$ and $n\gamma_j + k\delta_j > nk$. Let $m(f) = \mathrm{Min}\{\alpha_i + \beta_i : c_i \neq 0\}$ and $m(g) = \mathrm{Min}\{\gamma_j + \delta_j : d_j \neq 0\}$. If $f \approx g$, then $\{(\alpha_i, \beta_i) : \alpha_i + \beta_i = m(f)\} = \{(\gamma_j, \delta_j) : \gamma_j + \delta_j = m(g)\}$ as sets.*

Proof. By the similar method as in the proof of Theorem 3.2, we can prove it.

THEOREM 3.4. *Let $f = z^n + y^k + \Sigma c_i P_i$ where $n < k$, $(n,k) = 1$ and each c_i is a number if exists and $P_i = y^{\alpha_i} z^{\beta_i}$ with $n\alpha_i + k\beta_i > nk$ and $1 \leq \alpha_i \leq k-2$, $1 \leq \beta_i \leq n-2$. Then $f \approx z^n + y^k$ if and only if all c_i are zero.*

Proof. See [4] or use the similar technique as in the proof of Theorem 3.2.

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