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CHUNG–TEICHER TYPE STRONG LAW OF LARGE NUMBERS IN BANACH SPACE

SOO HAK SUNG

1. Introduction

Let $(B, \| \ \|)$ be a real separable Banach space. Let $\{X_n, n \geq 1\}$ be a sequence of Banach space valued random variables, and put $S_n = \sum_{i=1}^n X_i$.

Let ϕ be a positive, even and continuous function on R such that as $|x|$ increases,

$$\frac{\phi(x)}{x} \uparrow \quad and \quad \frac{\phi(x)}{x^2} \downarrow . \tag{1}$$

It was proved by Chung[4] that if $\{X_n, n \geq 1\}$ is a sequence of independent random variables with $EX_n = 0, n \geq 1$, and $\sum_{n=1}^\infty E\phi(|X_n|)/\phi(n) < \infty$, then $S_n/n \to 0$ a.s.. Teicher[6] proved that if $\{X_n, n \geq 1\}$ is a sequence of independent random variables with $EX_n = 0, n \geq 1$,

(i) $\sum_{i=2}^\infty (EX_i^2/i^4) \sum_{j=1}^{i-1} EX_j^2 < \infty$,
(ii) $\sum_{i=1}^n EX_i^2/n^2 \to 0$, and
(iii) $\sum_{n=1}^\infty P(|X_n| > a_n) < \infty$, for some positive constants $\{a_n\}$ with $\sum_{n=1}^\infty a_n^2 EX_n^2/n^4 < \infty$,

then $S_n/n \to 0$ a.s..

Woyczynski[7] proved Chung's strong law of large numbers(SLLN) in Banach spaces satisfying G_1-condition. Szynal and Kuczmaszewska [5] extended Teicher's SLLN in the case where $\{X_n, n \geq 1\}$ is a sequence of independent Hilbert space valued random variables. Choi and Sung([2],[3]) proved Chung's SLLN and Teicher's SLLN in Banach space under the assumption that the weak law of large numbers holds.

The purpose of this paper is to prove Chung-Teicher type SLLN in Banach space. Choi and Sung's results([2],[3]) follow from this result as corollaries. The method of our proof is similar to that of Choi and Sung[3].

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Throughout this paper, C will always stand for a positive constant which may be different in various places. I(A) means the indicator function of event A.

2. Main Result

To prove main theorem, we need the following lemma.

LEMMA 1. *(Yurinskii[8]). Let $X_1, \cdots, X_n$ be independent B-valued random variables with $E\|X_i\| < \infty (i = 1, \cdots, n)$. Let $\mathcal{F}_i$ be the σ-field generated by $(X_1, \cdots, X_i)$ for $i = 1, \cdots, n$, and let $\mathcal{F}_0$ be the trivial σ-field. Then, for $1 \le i \le n$,*

$$|E(\|S_n\||\mathcal{F}_i) - E(\|S_n\||\mathcal{F}_{i-1})| \le \|X_i\| + E\|X_i\|.$$

The proof of the following theorem is similar to that of Choi and Sung[3].

THEOREM 2. *Let $\{X_n, n \ge 1\}$ be a sequence of independent B-valued random variables, and let $\{a_n\}$ and $\{b_n\}$ be constants such that $0 < b_n \uparrow \infty$. Suppose that*

(i) $\sum_{i=2}^{\infty} \frac{a_i^2 E\phi(\|X_i\|)}{b_i^4 \phi(a_i)} \sum_{j=1}^{i-1} \frac{a_j^2 E\phi(\|X_j\|)}{\phi(a_j)} < \infty$,

(ii) $\frac{1}{b_n^2} \sum_{i=1}^{n} \frac{a_i^2 E\phi(\|X_i\|)}{\phi(a_i)} \to 0$,

(iii) $\sum_{i=1}^{\infty} P(\|X_i\| > a_i) < \infty$, *and*

(iv) $\sum_{i=1}^{\infty} \frac{a_i^4 E\phi(\|X_i\|)}{b_i^4 \phi(a_i)} < \infty$.

Then $S_n/b_n \to 0$ in probability iff $S_n/b_n \to 0$ a.s.

Proof. Assume that $S_n/b_n \to 0$ in probability. Define

$$X_i' = X_i I(\|X_i\| \le a_i), \quad S_n' = \sum_{i=1}^{n} X_i',$$

$$X_i'' = X_i I(\|X_i\| > a_i), \quad S_n'' = \sum_{i=1}^{n} X_i''.$$

From the condition (iii) and Borel-Cantelli lemma, we have $S_n''/b_n \to 0$ a.s.. Since $S_n/b_n \to 0$ in probability, we obtain

$$\frac{S_n'}{b_n} \to 0 \ \ in \ \ probability. \tag{2}$$

The first hypothesis of (1) implies that ϕ is non-decreasing function on $R^+ = [0, \infty]$. The second hypothesis of (1) implies that

$$\frac{x^2}{a_i^2} \leq \frac{\phi(|x|)}{\phi(a_i)} \quad for \quad |x| \leq a_i.$$

It follows that

$$E\|X_i'\|^2 \leq \frac{a_i^2 E\phi(\|X_i'\|)}{\phi(a_i)} \leq \frac{a_i^2 E\phi(\|X_i\|)}{\phi(a_i)}. \tag{3}$$

For each positive integers n and $i (1 \leq i \leq n)$, let

$$Y_{n,i} = E(\|S_n'\| | \mathcal{F}_i) - E(\|S_n'\| | \mathcal{F}_{i-1}),$$

where $\mathcal{F}_i = \sigma\{X_1', \cdots, X_i'\}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Then $\sum_{i=1}^n Y_{n,i} = \|S_n'\| - E\|S_n'\|$, and $|Y_{n,i}| \leq \|X_i'\| + E\|X_i'\|$ by Lemma 1. Hence we have by (3) and (ii) that

$$\begin{aligned}
&E\left|\frac{\|S_n'\| - E\|S_n'\|}{b_n}\right|^2 = \frac{1}{b_n^2} E\Big(\sum_{i=1}^n Y_{n,i}\Big)^2 = \frac{1}{b_n^2}\sum_{i=1}^n EY_{n,i}^2 \\
\leq &\frac{1}{b_n^2}\sum_{i=1}^n E(\|X_i'\| + E\|X_i'\|)^2 \leq \frac{4}{b_n^2}\sum_{i=1}^n E\|X_i'\|^2 \\
\leq &\frac{4}{b_n^2}\sum_{i=1}^n \frac{a_i^2 E\phi(\|X_i\|)}{\phi(a_i)} \to 0.
\end{aligned}$$

Thus $(\|S_n'\| - E\|S_n'\|)/b_n \to 0$ in probability. By (2), we have

$$\frac{E\|S_n'\|}{b_n} \to 0. \tag{4}$$

For each positive integer k, we define $m_k = inf\{n | b_n \geq 2^k\}$. The proof of Theorem will be completed if we show that $S_n'/b_n \to 0$ a.s.. First we show that

$$\frac{S_{m_k}'}{b_{m_k}} \to 0 \quad a.s. \tag{5}$$

By (4), it is enough to show that

$$\frac{\|S'_{m_k}\| - E\|S'_{m_k}\|}{b_{m_k}} \to 0 \quad a.s. \tag{6}$$

Consider the following simple identity.

$$(\|S'_{m_k}\| - E\|S'_{m_k}\|)^2 = (\sum_{i=1}^{m_k} Y_{m_k,i})^2 = \sum_{i=1}^{m_k} Y_{m_k,i}^2 + 2\sum_{i=2}^{m_k}\sum_{j=1}^{i-1} Y_{m_k,i}Y_{m_k,j}.$$

For the simplicity of notations, let

$$U_{m_k} = \sum_{i=2}^{m_k}\sum_{j=1}^{i-1} Y_{m_k,i}Y_{m_k,j}, \quad and \quad V_{m_k} = \sum_{i=1}^{m_k} Y_{m_k,i}^2.$$

To prove (6), it is enough to show that

$$\frac{U_{m_k}}{b_{m_k}^2} \to 0 \quad a.s., \tag{7}$$

and

$$\frac{V_{m_k}}{b_{m_k}^2} \to 0 \quad a.s.. \tag{8}$$

To prove (7), it suffices to show by Borel-Cantelli lemma that, for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P(|\frac{U_{m_k}}{b_{m_k}^2}| > \epsilon) < \infty.$$

Since $\{\sum_{j=1}^{i-1} Y_{m_k,i}Y_{m_k,j}, 2 \le i \le m_k\}$ and $\{Y_{m_k,i}, 1 \le i \le m_k\}$ are martingale differences for fixed m_k, we have

$$\sum_{k=1,m_k\neq m_{k+1}}^{\infty} P(|\frac{U_{m_k}}{b_{m_k}^2}|>\epsilon)\leq\frac{1}{\epsilon^2}\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}E(U_{m_k})^2$$

$$=\frac{1}{\epsilon^2}\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}\sum_{i=2}^{m_k}E(Y_{m_k,i}\sum_{j=1}^{i-1}Y_{m_k,j})^2$$

$$\leq\frac{1}{\epsilon^2}\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}\sum_{i=2}^{m_k}E\{(\|X_i'\|+E\|X_i'\|)^2(\sum_{j=1}^{i-1}Y_{m_k,j})^2\}$$

$$=\frac{1}{\epsilon^2}\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}\sum_{i=2}^{m_k}E(\|X_i'\|+E\|X_i'\|)^2E(\sum_{j=1}^{i-1}Y_{m_k,j})^2$$

$$=\frac{1}{\epsilon^2}\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}\sum_{i=2}^{m_k}E(\|X_i'\|+E\|X_i'\|)^2\sum_{j=1}^{i-1}E(Y_{m_k,j})^2$$

$$\leq C\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}\sum_{i=2}^{m_k}E\|X_i'\|^2\sum_{j=1}^{i-1}E\|X_j'\|^2$$

$$=C\sum_{i=2}^{\infty}E\|X_i'\|^2\sum_{j=1}^{i-1}E\|X_j'\|^2\sum_{\{k|m_k\geq i\}}\frac{1}{b_{m_k}^4},$$

where $\sum_{k=1,m_k\neq m_{k+1}}^{\infty}$ means that the summation is taken over all k such that $m_k\neq m_{k+1}$. From the definition of m_k, it follows that

$$\sum_{\{k|m_k\geq i\}}\frac{1}{b_{m_k}^4}\leq C\frac{1}{b_i^4}.$$

Hence we have by (i) and (3) that

$$\sum_{k=1,m_k\neq m_{k+1}}^{\infty} P(|\frac{U_{m_k}}{b_{m_k}^2}|>\epsilon)\leq C\sum_{i=2}^{\infty}E\|X_i'\|^2\sum_{j=1}^{i-1}\frac{E\|X_j'\|^2}{b_i^4}$$

$$\leq C\sum_{i=2}^{\infty}\frac{a_i^2E\phi(\|X_i\|)}{b_i^4\phi(a_i)}\sum_{j=1}^{i-1}\frac{a_j^2E\phi(\|X_j\|)}{\phi(a_j)}<\infty.$$

Thus (7) is proved. To prove (8), let

$$Z_{m_k,i} = Y^2_{m_k,i} - E(Y^2_{m_k,i}|\mathcal{F}_{i-1}), \quad 1 \le i \le m_k.$$

Then we obtain

$$\begin{aligned} EZ^2_{m_k,i} &= EY^4_{m_k,i} - E(E(Y^2_{m_k,i}|\mathcal{F}_{i-1}))^2 \\ &\le EY^4_{m_k,i} \le E(\|X'_i\| + E\|X'_i\|)^4 \le Ca_i^2 E\|X'_i\|^2. \end{aligned}$$

Thus we have by (iv) and (3) that

$$\begin{aligned} &\sum_{k=1, m_k \ne m_{k+1}}^{\infty} P\left(\left|\frac{\sum_{i=1}^{m_k} Z_{m_k,i}}{b^2_{m_k}}\right| > \epsilon\right) \le \frac{1}{\epsilon^2}\sum_{k=1}^{\infty}\frac{1}{b^4_{m_k}}E\left(\sum_{i=1}^{m_k} Z_{m_k,i}\right)^2 \\ &= \frac{1}{\epsilon^2}\sum_{k=1}^{\infty}\frac{1}{b^4_{m_k}}\sum_{i=1}^{m_k} E(Z^2_{m_k,i}) \le C\sum_{k=1}^{\infty}\frac{1}{b^4_{m_k}}\sum_{i=1}^{m_k} a_i^2 E\|X'_i\|^2 \\ &= C\sum_{i=1}^{\infty} a_i^2 E\|X'_i\|^2 \sum_{\{k|m_k \ge i\}}\frac{1}{b^4_{m_k}} \le C\sum_{i=1}^{\infty} a_i^2 E\|X'_i\|^2 \frac{1}{b_i^4} \\ &\le C\sum_{i=1}^{\infty}\frac{a_i^4 E\phi(\|X_i\|)}{b_i^4\phi(a_i)} < \infty. \end{aligned}$$

Applying Borel-Cantelli lemma, we obtain

$$\frac{\sum_{i=1}^{m_k} Y^2_{m_k,i} - \sum_{i=1}^{m_k} E(Y^2_{m_k,i}|\mathcal{F}_{i-1})}{b^2_{m_k}} \to 0 \quad a.s..$$

To finish the proof of (8), it suffices to show that

$$\frac{\sum_{i=1}^{m_k} E(Y^2_{m_k,i}|\mathcal{F}_{i-1})}{b^2_{m_k}} \to 0 \quad a.s..$$

By (ii) and (3), we have

$$\begin{aligned} &\frac{\sum_{i=1}^{m_k} E(Y^2_{m_k,i}|\mathcal{F}_{i-1})}{b^2_{m_k}} \le \frac{\sum_{i=1}^{m_k} E((\|X'_i\| + E\|X'_i\|)^2|\mathcal{F}_{i-1})}{b^2_{m_k}} \\ &= \frac{\sum_{i=1}^{m_k} E(\|X'_i\| + E\|X'_i\|)^2}{b^2_{m_k}} \le C\frac{\sum_{i=1}^{m_k} E\|X'_i\|^2}{b^2_{m_k}} \\ &\le C\frac{1}{b^2_{m_k}}\sum_{i=1}^{m_k}\frac{a_i^2 E\phi(\|X_i\|)}{\phi(a_i)} \to 0. \end{aligned}$$

Thus (5) is proved. By observing that for $m_k \le n < m_{k+1}$

$$\frac{\|S'_n\|}{b_n} \le \frac{\|S'_{m_k}\|}{b_{m_k}} + \max_{m_k \le n < m_{k+1}} \frac{\|S'_n - S'_{m_k}\|}{b_{m_k}},$$

we will obtain $S'_n/b_n \to 0$ a.s. if we show that

$$\max_{m_k \le n < m_{k+1}} \frac{\|S'_n - S'_{m_k}\|}{b_{m_k}} \to 0 \;\; a.s.. \tag{9}$$

If $m_k \le n < m_{k+1}$, then we have by (2) that

$$\begin{aligned}\frac{\|S'_{m_{k+1}-1} - S'_n\|}{b_{m_k}} &\le \frac{\|S'_{m_{k+1}-1}\|}{b_{m_k}} + \frac{\|S'_n\|}{b_{m_k}} \\ &\le 2\frac{\|S'_{m_{k+1}-1}\|}{b_{m_{k+1}-1}} + 2\frac{\|S'_n\|}{b_n} \to 0 \;\; in \;\; probability\end{aligned}$$

as $k \to \infty$. Hence there exists k_0 such that

$$\max_{m_k \le n < m_{k+1}} P\Big(\frac{\|S'_{m_{k+1}-1} - S'_n\|}{b_{m_k}} > \frac{\epsilon}{2}\Big) \le \frac{1}{2} \;\; for \;\; k \ge k_0.$$

From Skorokhod's inequality(see Breiman[1], p. 45), we have

$$\begin{aligned}&\sum_{k=k_0}^{\infty} P\Big(\max_{m_k \le n < m_{k+1}} \frac{\|S'_n - S'_{m_k}\|}{b_{m_k}} > \epsilon\Big) \\ \le &\sum_{k=k_0}^{\infty} \frac{P(\|S'_{m_{k+1}-1} - S'_{m_k}\| > \frac{\epsilon}{2} b_{m_k})}{1 - \max_{m_k \le n < m_{k+1}} P(\|S'_{m_{k+1}-1} - S_n\| > \frac{\epsilon}{2} b_{m_k})} \\ \le &2 \sum_{k=k_0}^{\infty} P(\|S'_{m_{k+1}-1} - S'_{m_k}\| > \frac{\epsilon}{2} b_{m_k}).\end{aligned} \tag{10}$$

By the same method of the proof of (5), it follows that

$$\frac{S'_{m_{k+1}-1}}{b_{m_k}} \to 0 \;\; a.s.$$

From this result, (5) and (10),

$$\sum_{k=k_0}^{\infty} P\Big(\max_{m_k \le n < m_{k+1}} \frac{\|S'_n - S'_{m_k}\|}{b_{m_k}} > \epsilon\Big) < \infty.$$

Applying Borel-Cantelli lemma, (9) is proved. Hence the proof of Theorem is completed.

REMARK. Teicher's SLLN(Choi and Sung[3]) for Banach space valued random variables follows from Theorem 2 letting $\phi(x) = x^2$ and $b_n = n$.

COROLLARY 3. *(Choi and Sung [2]). Let $\{X_n, n \geq 1\}$ be a sequence of independent B-valued random variables and let $\{b_n\}$ be constants such that $0 < b_n \uparrow \infty$. Assume that*

$$\sum_{n=1}^{\infty} \frac{E\phi(\|X_n\|)}{\phi(b_n)} < \infty. \tag{11}$$

Then $S_n/b_n \to 0$ in probability iff $S_n/b_n \to 0$ a.s..

Proof. Let $a_n = b_n$. The conditions (i) and (iv) of Theorem 2 are easily satisfied. The first hypothesis of (1) implies ϕ is non-decreasing function on $R^+ = [0, \infty]$. Hence we have by (11) and Markov's inequality that

$$\sum_{i=1}^{\infty} P(\|X_i\| > a_i) \leq \sum_{i=1}^{\infty} P(\phi(\|X_i\|) \geq \phi(a_i)) \leq \sum_{i=1}^{\infty} \frac{E\phi(\|X_i\|)}{\phi(a_i)} < \infty.$$

Thus the condition (iii) of Theorem 2 is satisfied. From (11) and Kronecker lemma,

$$\frac{1}{b_n^2} \sum_{i=1}^{n} \frac{b_i^2 E\phi(\|X_i\|)}{\phi(b_i)} \to 0 \quad a.s.$$

Thus the condition (ii) of Theorem 2 is satisfied. Therefore the assertion follows from Theorem 2.

REMARK. Under the same assumptions of Corollary 3, Choi and Sung[2] proved that $S_n/b_n \to 0$ in probability iff $S_n/b_n \to 0$ a.s. iff $S_n/b_n \to 0$ in L^1. But, we can not obtain from Theorem 2 that $S_n/b_n \to 0$ a.s. implies $S_n/b_n \to 0$ in L^1(see the following example).

EXAMPLE. Let (X_n) be independent random variables with

$$X_n = \begin{cases} n^2 \lg n & \text{with probability } \frac{1}{n(\lg n)^2} \\ \frac{1}{\lg n} & \text{with probability } 1 - \frac{1}{n(\lg n)^2}, \end{cases}$$

where $\lg x = \max\{1, \log^+ x\}$. Then $EX_n \sim \frac{n}{\log n}$. Take $\phi(x) = |x|, a_n = 1$ and $b_n = n$. From a simple calculation, the conditions (i), (ii), (iii) and (iv) of Theorem 2 are satisfied. Now we show that $S_n/n \to 0$ a.s.. Define $X_i' = X_i I(|X_i| \le 1)$, $X_i'' = X_i I(|X_i| > 1), S_n' = \sum_{i=1}^n X_i'$, and $S_n'' = \sum_{i=1}^n X_i''$.

$$\sum_{n=1}^{\infty} P(|X_n| > 1) \le C \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.$$

By the Borel-Cantelli lemma, we have $S_n''/n \to 0$ a.s.. $S_n'/n \to 0$ in probability follows from the following calculation:

$$E|\frac{S_n'}{n}| = \frac{\sum_{i=1}^n EX_i'}{n} \sim \frac{1}{n}\sum_{i=1}^n \frac{1}{\lg i}(1 - \frac{1}{i(\lg i)^2}) \sim \frac{1}{n}\sum_{i=1}^n \frac{1}{\lg i} \sim \frac{1}{\log n} \to 0$$

Thus $S_n/n \to 0$ in probability. Hence we have by Theorem 2 that $S_n/n \to 0$ a.s.. But, S_n/n does not converge to 0 in L^1, since

$$E|\frac{S_n}{n}| = \frac{1}{n}\sum_{i=1}^n EX_i \sim \frac{1}{n}\sum_{i=1}^n \frac{i}{\lg i} \sim \frac{n}{\log n} \to \infty.$$

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Department of Applied Mathematics
Pai Chai University
Taejon 302-735, Korea