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# UNIQUENESS FOR THE CAUCHY PROBLEM OF THE HEAT EQUATION WITHOUT UNIFORM CONDITION ON TIME

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## 1. Introduction

In the theory of heat conduction the temperature of an infinite rod is not always uniquely determined by its initial temperature. The following famous example

$$u(x,t) = \sum_{n=0}^{\infty} f^{(n)}(t)x^{2n}/(2n)! \tag{1.1}$$

where

$$f(t) = \begin{cases} e^{-1/t^2} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

satisfies the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

for $t > 0$, but $u(x, 0+) = 0$ for $-\infty < x < \infty$.

Also, in [CK] we have constructed a continuous function $u$ on $\mathbf{R}^n \times [0,T)$ satisfying $|u(x,t)| \leq C \exp(\epsilon/t)$ as an example of nonuniqueness for the Cauchy problem of the heat equation.

On the other hand there is a typical uniqueness theorem for the Cauchy problem of the heat equation as follows :

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THEOREM A([W]). *Let $u(x,t)$ be a continuous function on $\mathbf{R}^n \times [0,T]$ satisfying that*

$$(\partial_t - \Delta)u(x,t) = 0 \quad on \quad \mathbf{R}^n \times (0,T)$$

*and for some $C > 0$ and $a > 0$*

$$|u(x,t)| \le Ce^{a|x|^2} \quad on \quad \mathbf{R}^n \times [0,T]. \tag{1.2}$$

*Then $u(x,0) = 0$ on $\mathbf{R}^n$ implies that $u(x,t) \equiv 0$ on $\mathbf{R}^n \times [0,T]$.*

THEOREM B([F]). *Let $u(x,t)$ be a continuous function on $\mathbf{R}^n \times [0,T]$ satisfying that*

$$(\partial_t - \Delta)u(x,t) = 0 \quad on \quad \mathbf{R}^n \times (0,T)$$

*and*

$$\int_0^T \int_{\mathbf{R}^n} |u(x,t)|e^{-a|x|^2}\,dx\,dt \tag{1.3}$$

*is finite for some $a > 0$. Then $u(x,0) = 0$ on $\mathbf{R}^n$ implies that $u(x,t) \equiv 0$ on $\mathbf{R}^n \times [0,T]$.*

The theorem B is a little stronger than Theorem A. Note that the growth condition (1.2) or (1.3) is quite unrestricted with respect to the $x$ variable, but too restricted with respect to the $t$ variable to apply this theorem in many cases (see [KCK], for example).

In this paper we prove a more generalized uniqueness theorem of Cauchy problem under the following weaker growth condition

$$|u(x,t)| \le C \exp k(|x|^2 + 1/t), \quad t > 0$$

for some constants $C > 0$ and $k > 0$, instead of (1.2) and (1.3). Moreover, the growth condition does not require the continuity of $u(x,t)$ on $t = 0$.

## 2. Uniqueness Theorems

We first introduce the following function space to give a more generalized uniqueness theorem for the Cauchy problem than Theorem B.

DEFINITION 2.1. We denote by $\mathcal{E}(k)$ the set of all infinitely differentiable functions $\phi$ in $\mathbf{R}^n$ such that for any $h > 0$

$$|\phi|_{\mathcal{E}(k),h} = \sup_{\substack{x \in \mathbf{R}^n \\ \alpha}} \frac{|\partial^\alpha \phi(x)| \exp k|x|^2}{h^{|\alpha|}\alpha!} \tag{2.1}$$

is finite. The topology in $\mathcal{E}(k)$, defined by the above seminorms, makes $\mathcal{E}(k)$ a $FS$-space. In fact, it is the projective limit topology over all $h > 0$. We denote by $\mathcal{E}'(k)$ the strong dual of the space $\mathcal{E}(k)$.

LEMMA 2.2. *Let $P(\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha \partial^\alpha$ be a differential operator of infinite order with constant coefficients satisfying that there exist constants $L > 0$ and $C > 0$ such that*

$$|a_\alpha| \le CL^{|\alpha|}/\alpha! \tag{2.2}$$

*for all $\alpha$. Then the operators*

$$P(\partial) : \mathcal{E}(k) \to \mathcal{E}(k) \tag{2.3}$$

*and*

$$P(\partial) : \mathcal{E}'(k) \to \mathcal{E}'(k) \tag{2.4}$$

*are continuous.*

*Proof.* Let $\phi$ belong to $\mathcal{E}(k)$ and $h > 0$. Then it follows that

$$\begin{aligned}
&|\partial^\beta P(\partial)\phi(x)| \exp k|x|^2 \\
&\le \sum_{|\alpha|=0}^{\infty} |a_\alpha|\, |\partial^{\alpha+\beta}\phi(x)| \exp k|x|^2 \\
&\le \sum_{|\alpha|=0}^{\infty} \frac{CL^{|\alpha|}}{\alpha!} |\phi|_{\mathcal{E}(k),h} h^{|\alpha+\beta|}(\alpha+\beta)! \\
&\le C|\phi|_{\mathcal{E}(k),h}(2h)^{|\beta|}\beta! \sum_{|\alpha|=0}^{\infty} (2hL)^{|\alpha|}.
\end{aligned}$$

Thus, for any $H > 0$ if we choose $h > 0$ so small that $2Lh < 1/2$ and $2h < H$ then we obtain

$$|P(\partial)\phi(x)|_{\mathcal{E}(k),H} \le C|\phi|_{\mathcal{E}(k),h}, \quad \phi \in \mathcal{E}(k), \tag{2.5}$$

which proves that (2.3) is continuous. Also the continuity of (2.4) is easily obtained from this.

From now on we denote by $E(x,t)$ the $n$-dimensional heat kernel :

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2}\exp[-|x|^2/4t], & t > 0 \\ 0 & t \leq 0. \end{cases}$$

PROPOSITION 2.3. *Let $g(x)$ be a continuous function satisfying that for some constants $C > 0$ and $k > 0$*

$$|g(x)| \leq C \exp k|x|^2, \quad x \in \mathbf{R}^n, \tag{2.6}$$

*and $G(x,t) = g(x) * E(x,t)$ where $*$ denotes the convolution with respect to the $x$ variable. Then $G(x,t)$ is a well defined $C^\infty$-function in $\mathbf{R}^n \times (0, 1/4k)$ and satisfies that*

$$\begin{aligned} &\text{(i) } (\partial_t - \Delta)G(x,t) = 0, \quad 0 < t < 1/4k \\ &\text{(ii) } |G(x,t)| \leq C\exp(2k|x|^2), \quad 0 < t < 1/16k \\ &\text{(iii) } G(x,t) \to g(x) \textit{ locally uniformly on } \mathbf{R}^n \textit{ as } t \to 0+. \end{aligned} \tag{2.7}$$

*Proof.* Since $t - 1/4k < 0$, $G(x,t)$ is well defined and satisfies the heat equation. For convenience we only consider the 1-dimensional case. For $0 < t < 1/16k$,

$$\begin{aligned} G(x,t) &= \int_{\mathbf{R}} E(y,t)g(x-y)\,dy \\ &= \frac{1}{\sqrt{\pi}}\int_{\mathbf{R}} e^{-r^2} g(x - \sqrt{4t}\,r)\,dr \\ &\leq \frac{1}{\sqrt{\pi}}\int_{\mathbf{R}} e^{-r^2} e^{2k(x^2+4tr^2)}\,dr \\ &\leq \frac{e^{2kx^2}}{\sqrt{\pi}}\int e^{(8kt-1)r^2}\,dr \\ &= e^{2kx^2}/\sqrt{1-8kt} \;\leq\; \sqrt{2}e^{2k|x|^2}, \end{aligned}$$

which proves (ii). Let $\delta > 0$ and $0 < t < 1/4k$. Then it follows from the property of the heat kernel $E(x,t)$ that for some $A > 0$ and $0 < t < T < 1/4k$

$$\begin{aligned}\int_{|y|\geq\delta} E(y,t)e^{ky^2}\,dy &= \frac{1}{\sqrt{4\pi t}}\int_{|y|\geq\delta} e^{(k-1/4t)y^2}\,dy \\ &\leq \frac{1}{\sqrt{4A\pi t}}\int_{|y|\geq\delta/\sqrt{A}} e^{-y^2/4t}\,dy \to 0 \quad \text{as } t\to 0+\,. \end{aligned} \tag{2.8}$$

On the other hand, for $0 < t < T$

$$\begin{aligned}|G(x,t)-g(x)| &\leq \int E(z-x,t)|g(z)-g(x)|\,dz \\ &= \int_{|z-x|<\delta} E(z-x,t)|g(x)-g(z)|\,dz \\ &\quad + \int_{|z-x|\geq\delta} E(z-x,t)|g(x)-g(z)|\,dz \\ &= I_1 + I_2.\end{aligned}$$

Let $K$ be a compact subset of $\mathbf{R}$. Since $g(x)$ is uniformly continuous on a $\delta$-neighborhood $K_\delta$ of $K$ it follows that for any $\varepsilon > 0$ there exists a constant $\delta > 0$ such that $|z-x| < \delta$ implies $|g(x)-g(z)| < \varepsilon$ for $x, z \in K_\delta$. Then we obtain from these facts that $|I_1| \leq \varepsilon$ for all $x \in K$. Also it follows from (2.8) that for every $x \in K$

$$\begin{aligned}|I_2| &= \int_{|y|\geq\delta} E(y,t)|g(y-x)-g(x)|\,dy \\ &\leq \int_{|y|\geq\delta} E(y,t)e^{2k(x^2+y^2)}e^{kx^2}\,dy \\ &\leq C(K)\int_{|y|\geq\delta} E(y,t)e^{2ky^2}\,dy \to 0 \text{ as } t\to 0+\end{aligned}$$

where $C(K)$ is a constant depending on $K$. This completes the proof.

The following lemma is very useful later. For the details of the proof we refer to Komatsu [K] :

LEMMA 2.4. *For any $L > 0$ and $\varepsilon > 0$ there exist a function $\gamma(t) \in C_0^\infty(\mathbf{R})$ and a differential operator $P(d/dt)$ of infinite order such that*

$$\begin{aligned} &supp\, \gamma \subset [0, \varepsilon], \ |\gamma(t)| \le C \exp(-L/t), \quad 0 < t < \infty; \\ &P(d/dt) = \sum_{k=0}^{\infty} a_k (d/dt)^k, \ \ |a_k| \le C_1 L_1^k / k!^2, \quad 0 < L_1 < L; \\ &P(d/dt)\gamma(t) = \delta + w(t) \end{aligned} \tag{2.9}$$

*where $w \in C_0^\infty(\mathbf{R})$ with $supp\, w \subset [\varepsilon/2, \varepsilon]$ and $\delta$ is a Dirac measure.*

We are now in a position to state and prove the main theorem in this paper.

THEOREM 2.5. *Let $u(x,t)$ be a function on $\mathbf{R}^n \times (0,T)$ satisfying that*

(i) $(\partial_t - \Delta)u(x,t) = 0, \quad 0 < t < T,$

(ii) *For some $k > 0$, there exists $C > 0$ such that*

$$|u(x,t)| \le C \exp k(|x|^2 + 1/t), \quad 0 < t < T,$$

(iii) $\lim_{t \to 0+} \int u(x,t)\phi(x)\, dx = 0$ *for every $\phi \in \mathcal{E}(2k)$.*

*Then $u(x,t)$ is identically zero on $\mathbf{R}^n \times [0,T]$. Here $T$ may be $\infty$.*

*Proof.* In view of Theorem A or Theorem B in the introduction we have only to show that $u(x,t)$ is identically zero on $\mathbf{R}^n \times [0, T_0]$ for sufficiently small $T_0 > 0$.

Now choose a function $v, w$ and a differential operator $P(d/dt)$ of infinite order as in Lemma 2.4. Let

$$\tilde{u}(x,t) = \int_0^T u(x, t+s)\gamma(s)\, ds,$$

for $0 < t < T_0$. Then taking $\varepsilon = T_0$, $2T_0 < \min(T, 1/16k)$, and $L > k$ in Lemma 2.4 and using the condition (ii) we have

$$|\tilde{u}(x,t)| \le CT \exp k|x|^2, \quad 0 \le t \le T_0.$$

Therefore, $\tilde{u}(x,t)$ is a continuous function on $0 \le t \le T_0$.

Moreover, $\tilde{u}$ satisfies

$$(\partial_t - \Delta)\tilde{u}(x,t) = 0 \quad 0 < t \leq T_0.$$

Then it follows that for $0 < t < T_0$

$$P(-\Delta)\tilde{u}(x,t) = u(x,t) + \int_0^T u(x,t+s)w(s)\,ds \tag{2.10}$$

Now let

$$v(x,t) = -\int_0^T u(x,t+s)w(s)\,ds$$

for $0 < t < T_0$. Then $v(x,t)$ also satisfies the heat equation and satisfies that

$$u(x,t) = P(-\Delta)\tilde{u}(x,t) + v(x,t). \tag{2.11}$$

Also, for some $C' > 0$ and $C'' > 0$ we obtain

$$\begin{aligned} |v(x,t)| &\leq C\int_0^T \exp k(|x|^2 + \frac{1}{t+s})|w(s)|\,ds \\ &\leq C' \int_{T_0/2}^{T_0} \exp k(|x|^2 + \frac{1}{t+s})\,ds \\ &\leq C'' \exp k|x|^2, \end{aligned}$$

which means that $v(x,t)$ is also a continuous function on $\mathbf{R}^n \times [0, T_0]$ with the same growth type as $\tilde{u}(x,t)$. Let $g(x) = \tilde{u}(x,0)$ and $h(x) = v(x,0)$. Then it follows that $g(x)$ and $h(x)$ are continuous functions on $\mathbf{R}^n$ satisfying that for some $C > 0$,

$$|g(x)| \leq C\exp k|x|^2, \quad |h(x)| \leq C\exp k|x|^2, \quad x \in \mathbf{R}^n. \tag{2.12}$$

From these facts, we see that $g$ and $h$ belong to $\mathcal{E}'(2k)$. For the differential operator $P(d/dt)$, Lemma 2.2 and (2.9) imply that $P(-\Delta) : \mathcal{E}'(2k) \to \mathcal{E}'(2k)$ is continuous. We define $P(-\Delta)g + h \in \mathcal{E}'(2k)$ by

$$[P(-\Delta)g + h](\phi) = \int g(x)P(\Delta)\phi(x)\,dx + \int h(x)\phi(x)\,dx \tag{2.13}$$

for every $\phi \in \mathcal{E}(2k)$. Since $P(\Delta) : \mathcal{E}(2k) \to \mathcal{E}(2k)$ is continuous this is well defined. Then combining the Lebesgue dominated convergence theorem and the initial condition (ii) we obtain that for every $\phi \in \mathcal{E}(2k)$

$$\begin{aligned} & [P(-\Delta)g + h](\phi) \\ &= \lim_{t \to 0+} [\int \tilde{u}(x,t) P(\Delta)\phi(x)\, dx + \int v(x,t)\phi(x)\, dx] \\ &= \lim_{t \to 0+} \int [P(-\Delta)\tilde{u}(x,t) + v(x,t)]\phi(x)\, dx \\ &= \lim_{t \to 0+} \int u(x,t)\phi(x)\, dx \\ &= 0 \end{aligned} \tag{2.14}$$

Now let $a(x,t) = g(x) * E(x,t)$ and $b(x,t) = h(x) * E(x,t)$ for $0 < t < T_0$, where $*$ denotes the convolution with respect to the $x$ variable. Then by Proposition 2.3 $a(x,t)$ and $b(x,t)$ satisfy (i)$\sim$ (iii) of (2.7). Putting $A(x,t) = \tilde{u}(x,t) - a(x,t)$ and $B(x,t) = v(x,t) - b(x,t)$ we obtain that $A(x,t)$ and $B(x,t)$ satisfy the hypothesis of Theorem A in Introduction. Therefore,

$$\tilde{u}(x,t) = g(x) * E(x,t)$$

and

$$v(x,t) = h(x) * E(x,t).$$

It follows from (2.11) and (2.13) that for $0 < t < T_0$

$$\begin{aligned} u(x,t) &= P(-\Delta)\tilde{u}(x,t) + v(x,t) \\ &= P(-\Delta)(g * E) + h * E \\ &= [P(-\Delta)g + h] * E \\ &= [P(-\Delta)g + h](E(x-y,t)) \\ &= 0, \end{aligned}$$

since $E(\cdot - y, t)$ belongs to $\mathcal{E}(2k)$ for each $y$ and for each $0 < t < T_0$. This completes the proof.

REMARK. (i) In the condition of the above theorem, the continuity of $u(x,t)$ on $\mathbf{R}^n \times [0,T]$ is not required. Thus this uniqueness theorem is a little bit stronger than already known theorems.

(ii) It is easily seen that this theorem generalizes Theorem A.

(iii) The initial condition (iii) of this theorem is, more or less, unsatisfactory. But in view of the example as seen in [CK], it can be regarded as an optimal one. The space $\mathcal{E}(2k)$ can be replaced by $\mathcal{E}(k')$ for some $k' > k$. Also, it can be weakened as follows:

$$\lim_{t\to 0+} \int u(x,t)e^{-k|x|^2}\, dx = 0.$$

Finally, we give here the uniqueness theorem of temperature function on the semi-infinite rod.

THEOREM 2.6. *Let $u(x,t)$ be a function on $[0,\infty)\times(0,T)$ satisfying that*

(i) $(\partial_t - \Delta)u(x,t) = 0,\ 0 < x,\ 0 < t < T,$

(ii) *For some $k > 0$, there exists $C > 0$ such that*

$$|u(x,t)| \le C \exp k(x^2 + 1/t),\ 0 < x,\ 0 < t < T,$$

(iii) $\lim_{t\to 0+} \int_0^\infty u(x,t)\phi(x)\,dx = 0$ *for every $\phi \in \mathcal{E}(2k)$.*

(iv) $u(0,t) = 0,\ 0 < t < T.$

*Then $u(x,t)$ is identically zero on $\mathbf{R} \times [0,T]$. Here $T$ may be $\infty$.*

*Proof.* Define $v(x,t)$ on $\mathbf{R}$

$$v(x,t) = \begin{cases} u(x,t), & x \ge 0 \\ -u(-x,t), & x < 0 \end{cases}$$

Then the reflection principle of the heat solution (see [W], p.115) implies that $v(x,t)$ is also a solution of the heat equation and satisfies the hypothesis (i)~(iii) of Theorem 2.5. Therefore $v(x,t)$ must be identically zero, which completes the proof.

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