J. Korean Math. Soc. 31 (1994), No. 2, pp. 213–239

AN INDUCED MIXING FLOW UNDER 1 AND α

KYEWON K. PARK

I. Introduction

W.Ambrose has shown that an ergodic flow (R-action) can be represented as a flow built under a function (called a ceiling function) with a transformation (Z-action) on a base [1]. The σ-algebra of measurable sets in the representation is the completed product σ-algebra of measurable sets in the base and the Lebesgue sets in the time axis. D.Rudolph has improved this representation theorem so that the ceiling function takes only two irrationally related values, say α_1 and α_2 [10]. In fact, in the case of a flow of positive entropy, the proof of Rudolph's representation theorem is quite easy if we make use of the powerful theorems that every flow of positive entropy has a Bernoulli factor of full entropy and that a Bernoulli flow is isomorphic to a Totoki flow (a step coded flow with an independent partition on a base [12]) [7].

When a flow is represented under a function with finitely many values, the base has an obvious partition according to the values of the function. The significant part of Rudolph's representation is the fact that the obvious partition of the base can be constructed so that it generates the σ-algebra of the base. Hence the "corresponding" partition of the flow generates the σ-algebra of the flow. Rudolph's theorem turned out to be useful in extending some of the properties of Z-actions to those of R-actions [7]. We mention that the set of step coded flows is a $\bar{d}$-dense subset of all ergodic flows [8]. Properties of the flows of this type in terms of the values of ceiling functions and properties of their base transformations have been investigated in [8].

Throughout this paper we let $(\Omega, S_t, \mathcal{F}, \nu)$ be an ergodic flow represented under a function. We denote the base of the flow by (X, T, β, μ).

Received February 13, 1993.

This research has been supported in part by National Science Foundation Grant DMS 8604202, GARC-KOSEF and Ajou University .

If the flow $(\Omega, S_t, \mathcal{F}, \nu)$ is ergodic, then so is the base (X, T, β, μ). We sometimes denote the flow by $[X, T, \beta, \mu, f]$ where f is a ceiling function. In a two step coded case, since only the ratio between two values matters (modulo time scaling), we can assume that the larger value is 1.

Let $(\Omega, S_t, \mathcal{F}, \nu)$ denote an ergodic flow which has non-ergodic time action $S_{1-\alpha}$. We make use of Rudolph's theorem to represent this flow under a function of values 1 and α. We denote the new base by (X, T, β, μ). We claim that none of the flows $[X, T, \beta, \mu, f]$, where f takes values of either 1 or α, has an ergodic action $S_{1-\alpha}$. We see this as follows:

Since $S_{1-\alpha}$ is not an ergodic action in $(\Omega, \mathcal{F}, \nu)$, there exists an invariant function $g(w)$ under $S_{1-\alpha}$ where $w = (x, t)$ for $x \in X$, which means $g(w)$ is a periodic function with period $1 - \alpha$. That is, we have $g(w) = g(S_{1-\alpha}(w)) = g(w + (1 - \alpha))$ where $w + (1 - \alpha)$ denotes $(x, t + (1 - \alpha))$ with the identification $(x, f(x)) = (Tx, 0)$. If we let $\Omega_0 = \{(x, t) : x \in X,\ 0 \leq t < \alpha\}$, then clearly $g(w)$ restricted to Ω_0 is a periodic function with period $1 - \alpha$. Let $[X, T, \beta, \mu, f] = (\Omega', S'_t, \mathcal{F}', \nu')$ be a new flow where $f(x)$ is either 1 or α. We define

$$g'(w) = g'(x, t) = \begin{cases} g(x, t) & \text{if } 0 \leq t < \alpha \\ g(x, t - (1 - \alpha)) & \text{if } \alpha \leq t < 1 \end{cases}$$

Informally, $g'(w)$ is an extension of g on Ω_0 to Ω'. By the definition of g', it is clear that $g'(w)$ is invariant under $S'_{1-\alpha}$. Hence the flow $(\Omega', S'_t, \mathcal{F}', \nu')$ has the non-ergodic time action $S'_{1-\alpha}$.

Since all flows in the Kakutani equivalence class of $(\Omega, S_t, \mathcal{F}, \nu)$ can be built with the base (X, T, β, μ), we consider a mixing flow in the class built with the base. By the argument given above, this mixing flow can not be represented under a function of 1 and α with a base isomorphic to (X, T, β, μ). However, it is not yet clear if it is possible to represent the flow under 1 and π for some irrational π (different from α) with a base isomorphic to (X, T, β, μ).

It is shown that, unlike two step coded representations, a mixing flow $[X, T, \beta, \mu, f]$ can be built under a function of three values with a base isomorphic to (X, T, β, μ) [See 9]. This in turn implies that the properties of a base transformation do not tell us more about the properties of a three step coded flow over the base. It is proven in

[9] that there exist two non-isomorphic flows (one Bernoulli and the other non-mixing) over the same base partition, each of which is built under a function whose values are constant on each of the atoms of the partition and are independent over the rationals.

We are going to show that if a flow built under 1 and α (we will call the flow $\{1,\alpha\}$-flow) with a base (X,T,β,μ) has the ergodic time action $S_{1-\alpha}$, then there exists a mixing $\{1,\alpha\}$-flow with an isomorphic base. (By the argument given above, it is clear that the time $1-\alpha$ action on any $\{1,\alpha\}$-flow with the base (X,T,β,μ) is ergodic.) We accomplish this by inducing the flow in a special way. Furthermore, it is not hard to see from out construction that partitions according to the values 1 and α which give rise to mixing flows form a dense collection of partitions of two sets.

We may mention a couple of results in this direction. A.Katok has shown that if a base is an interval exchange map and the ceiling function is of bounded variation, then the flow is not mixing [5]. It is known that the $\{1,\alpha\}$-flow with the obvious partition of the Chacon transformation on the base is weakly mixing but not mixing [2]. Moreover, we can use the same argument to prove that in the case of the Chacon transformation on the base, if the set $P_1 = \{x \in X : f(x) = 1\}$ is a finite union of level sets at some stage, then the flow is weakly mixing but not (stronly) mixing. Since finite unions of level sets form an algebra generating the σ-algebra on the base, we can say that $\{1,\alpha\}$-flows are weakly mixing for a dense collection of partitions of two sets.

We define (X,T,β,μ) and (X',T',β',μ') to be related if and only if there exists a flow which can be represented under 1 and α with each of these as a base. There is an elementary proof that this is an equivalence relation [11]. It is also a restricted orbit equivalence as defined in [13]. As described in a paragraph above, each Kakutani equivalence class is divided into at least two different classes in this restricted sense. In fact, it is recently proved that each Kakutani equivalence class except the Loosely Bernoulli class is divided into uncountably many different α-equivalence classes [3]. Also residual properties of ergodicity, weakly mixing, k-mixing and Bernoulliness are investigated in [4].

I would like to thank Professors D.Ornstein and D.Rudolph for helpful discussions about many aspects of this paper.

II. Construction of a mixing flow $(\tilde{\Omega}, \tilde{S}_t, \tilde{\mathcal{F}}, \tilde{\nu})$

II.1 Preliminaries

We now assume that $(\Omega, S_t, \mathcal{F}, \nu)$ is an ergodic flow built under a function f with values 1 and α, unless otherwise stated. It is known that if a flow S_t is ergodic, then the time t_0 transformation S_{t_0} is ergodic except for countably many $t_0's$. It is not hard to see that if a partition $P = \{P_1, \dots P_\ell\}$ generates the σ-algebra of the base under T and S_{t_0} is ergodic, then the corresponding flow partition $\overline{P} = \{\overline{P}_1, \dots, \overline{P}_\ell\}$ where $\overline{P}_i = \{(x,t) : x \in P_i,\ 0 \le t < f(x)\}$ generates the σ-algebra of the flow under S_{t_0}.

Given a partition $Q = \{Q_1, \dots, Q_\ell\}$ of Ω, we define a flow Q-name of a point w from time 0 to time s to be an interval partitioned in such a way that $t \in [0, s]$ is in Q_i if $S_t(w) \in Q_i$. For a given t_0, the discrete Q-name of a point w under the action S_{t_0} from 0 to k is analogously defined. If the partition under consideration is clear in the context, then we say the name of a point instead of Q-name of the point.

We assume that $S_{1-\alpha}$, denoted by S in this section, is ergodic. We will change the ceiling function so that S_t is mixing. That is, we change the partition of the base according to the values 1 and α.

For technical convenience we assume the following:

(i) $1 - \alpha < \alpha$

(ii) $\mu(P_1) > 1/2$ where $P_1 = \{x \in X : f(x) = 1\}$.

We denote the flow orbit of a point w by $O(w)$. There is a natural identification between $O(w)$ and R. We denote the largest integer $\le t$ by $[t]$.

The crux of the argument is the observation that if we remove an interval whose length is a multiple of $1-\alpha$ from a flow orbit $O(w)$, then the induced discrete orbit of the point under S is a subset of the orbit of the point under S before the removal of the interval. We note that if we remove an interval whose length is not a multiple of length $1-\alpha$, then the induced orbit of the point under S is completely different from the orbit of the point under S before the removal.

By the definition of a mixing flow, it is enough to make the transformation S mixing. We accomplish this by constructing a sequence of flows $S_t^{(i)}$ built under 1 and α with isomorphic bases such that $S_{1-\alpha}^{(i)}{}'s$

become successively closer to having the mixing property. Hence the limit of these flows is mixing.

In the case of a transformation, N.Friedman and D.Ornstein have proved that any ergodic transformation can be induced so that the induced transformation has the mixing property [4]. We use their method to construct an induced flow $(\tilde{\Omega}, \tilde{S}_t, \tilde{\mathcal{F}}, \tilde{\nu})$ under 1 and α with a base $(\tilde{X}, \tilde{T}, \tilde{\beta}, \tilde{\mu})$ isomorphic to (X, T, β, μ). Hence we assume that readers have some familiarity with their ideas. Since the more important issue here is to construct the base $(\tilde{X}, \tilde{T}, \tilde{\beta}, \tilde{\mu})$ which is isomorphic to the original base, we will describe in detail the construction part, highlighting the differences between our situation of a continuous action and a discrete action. We will omit the proof of the mixing property of $S_{1-\alpha}$ of the limit flow since it is the same as in the discrete case [4].

If $R = \{R_1, \ldots, R_\ell\}$ and $Q = \{Q_1, \ldots, Q_m\}$ are partitions of a flow, then $R \vee Q$ denotes the partition whose atoms are $\{R_i \cap Q_j : i = 1, \ldots, \ell,\ j = 1, \ldots, m\}$. Let $P = \{P_0, P_1\}$ be the partition of X according to the values of f. We fix an increasing sequence of finite partitions $P = P^1 \subset P^2 \subset P^3 \subset \cdots$ of X so that $V_{i=1}^{\infty} P^i$ generates the σ-algebra β.

We define a set $A (\subset \Omega)$ to be $\varepsilon - Q$ (independent) if $|\nu(A \cap Q_i) - \nu(A)\nu(Q_i)| < \varepsilon\nu(A)$ for all $i = 1, \ldots, m$. We say that the k-long names of a set A satisfy the ergodic theorem within ε with respect to Q if

$$|(1/k) \sum_{j=0}^{k-1} \chi_{Q_i}(S^j w) - \nu(Q_i)| < \varepsilon \text{ for a.e. } x \in A \text{ and all } i = 1, \ldots, m.$$

If the k-long names of the set A satisfy the ergodic theorem within ε/m, then we say that the set A has the property $E(S, k, Q, \varepsilon)$. We note that if a set A has the property $E(S, k, Q, \varepsilon)$, then the k-long names of the set A satisfy the ergodic theorem within ε for any union of the atoms in Q. We say that a set A has the property $E_r(S, k, Q, \varepsilon)$ if the k'-long names of the set A satisfies the ergodic theorem with respect to Q within ε/m for all $k' \geq k$. We also say that the ℓ-long flow names of a set A satisfy the ergodic theorem within ε with respect to Q if

$$|(1/\ell) \int_0^{\ell} \chi_{Q_i}(S_t w) - \nu(Q_i)| < \varepsilon \text{ for a.e. } x \in A \text{ and all } i = 1, \ldots, m.$$

Using a Rochlin tower of the base transformation, it is easy to construct a flow skyscraper of heights $\geq h$ for any given h. We denote the height of the skyscraper by $h(x)$ for every x in the base of the skyscraper. We can build a flow skyscraper of heights between h and $2h$ so that $h(x)$ takes only finitely many values each of which is a linear sum of $1's$ and $\alpha's$. Let B denote the base of the skyscraper and let B_0 be a subset of B. If B_0 has a unique flow name between time 0 and $h(x)$ for $x \in B_0$, then we call this subskyscraper with the base B_0 a column. Given s and s', we call the set $B_0[s, s') = \{(x, t) : x \in B_0,\ s \leq t < s'\}$ the $[s, s')$-level set. When s and s' need not be mentioned, then we simply call the set a level set. If $s = s'$, then we call the set a flat level set, or s-level set if s needs to be mentioned. The s-level set of the $[s, s')$-level set is called the base of the level set. We note that a flat level set has the same name under S up to the top of the skyscraper. When B_0' is a subset of B_0, the flow skyscraper with the base B_0' is called a subcolumn.

Let $Q = \{Q_1, \ldots, Q_m\}$ be a partition of a flow such that each Q_i is of the type $\{(x, t) : x \in C,\ t_i \leq t < t_i'\}$ where C is a measurable subset of X and t_i and t_i' are between 0 and 1. (We always assume that the flow partition Q is finer than $\overline{P}$.) In this case we say that the partition Q is rectangular. Note that if we have a rectangular partition, then we can build a flow skyscraper so that the skyscraper has finitely many columns according to their flow names. We call $\eta = \min\{t_i' - t_i : i = 1, \ldots, m\}$ the height of the partition Q. We define

$$\begin{aligned} Q_i^\delta = &\{(x, t) : x \in C,\ t_i \leq t < t_i + (\delta\eta/2)\} \cup \\ &\{(x, t) : x \in C,\ t_i' - (\delta\eta/2) \leq t < t_i'\} \\ &\text{for any}\quad 0 < \delta < 1. \end{aligned}$$

Let $\Theta = Q_1^\delta \cup Q_2^\delta \cup \cdots \cup Q_m^\delta$ and let Q' be the partition $\{\Theta, \Theta^C\}$. It is clear that $\nu(\Theta) < \delta$. We denote $Q \vee Q'$ by Q^δ.

LEMMA 1. *Given any $\varepsilon > 0$ and $\delta > 0$, if a t_1-level set, denoted by D, of a column has the property $E(S, k, Q^\delta, \varepsilon)$, then the $(t_1 - (\delta\eta/2), t_1 + (\delta\eta/2))$-level set, denoted by D^δ, has the property $E(S, k, Q, 2\varepsilon + \delta)$.*

Proof. Let $w = (x, t)$ be a point in D^δ. Let $w' = (x, t_1)$ be the point in D with the same x. We note that k-long Q-names of $w = \{w_i\}_{i=1}{}^k$

and $w' = \{w'_i\}_{i=1}{}^k$ differ at the indices only if w'_i belongs to Θ. Since k-long names of w' satisfies the ergodic theorem for Q^δ and $\nu(\Theta) < \delta$, this happens with frequency less than $\delta + \varepsilon$. Therefore, the k-long Q-name of w deffers from that of w' by at most $2\varepsilon + \delta$. Hence w has the property $E(S, k, Q, 2\varepsilon + \delta)$.

COROLLARY 1. *If none of the flat level sets in a (t_1, t_2)-level set has the property $E(S, k, Q, \varepsilon)$, then none of the flat level sets in $(t_1 - (\varepsilon\eta/4), t_2 + (\varepsilon\eta/4))$-level set has the property $E(S, k, Q^{\varepsilon/2}, \varepsilon/4)$.*

Proof. If there exists a t-level set with the property $E(S, k, Q^{\varepsilon/2}, \varepsilon/4)$ in the $(t_1 - (\varepsilon\eta/4), t_2 + (\varepsilon\eta/4))$-level set, then by Lemma 1, $(t - (\varepsilon\eta/4), t + (\varepsilon\eta/4))$-level set has the property $E(S, K, Q, \varepsilon)$.

Informally, what Corollary 1 says is that any "bad" level set is surrounded by a "bad" (in a more stringent sense) level set.

For any δ and ε, we let $L_1(\delta, \varepsilon)$ be an integer such that the subset $R_0(\subset \Omega)$ of points which do not have the property $E_r(S, L_1, Q^{\varepsilon/2}, \varepsilon/4)$ has measure less than $\delta/2$.

LEMMA 2. *Given any $\delta > 0$ and $\varepsilon > 0$, there exists a flow skyscraper such that it has the following property.*

> $(*)$ *Let C_i be a column according to $Q^{\varepsilon/2}$-names of the skyscraper and let $E_i = C_i \cap R_0$. Then $\nu(E_i)/\nu(C_i) < \delta$ for all i.*

Proof. Consider the set $A = \{(x, t) : x \in X,\ 0 \le t < 1 - \alpha\}$. There exists $\ell_0(\gg 1 - \alpha)$ such that the subset F of A which satisfies the flow ergodic theorem within $\delta/4$ with respect to $R = \{R_0, R_0{}^c\}$ for all $\ell \ge \ell_0$ has measure bigger than $(1/2)\nu(A)$. Let $w = (x_0, t_0) \in F$. Clearly given this x_0, $\{(x_0, t) : 0 \le t < 1 - \alpha\}$ satisfies the flow ergodic theorem within $\delta/2$ with respect to R for all $\ell > \ell_0$. Let $B' = \{x \in X : (x, t_0) \in F \text{ for some } t_0\}$. Let B be a subset of B' such that the flow skyscraper built with B as a base has heights bigger than ℓ_0. Hence each column of the flow skyscraper with the base B has the property $(*)$.

A couple of things need to be mentioned here. First, the partition of a skyscraper into columns according to Q^δ-names (for any δ) and according to Q-names are the same. Secondly, since any t_0-level set for $t_0 < h(x) - (1 - \alpha)L_1$ has the same L_1-long name under S, a t_0-level set is either completely contained in R_0 or ${R_0}^c$. Hence the set E_i, except for its subset contained in $\{(x,t) : x \in B, h(x)-(1-\alpha)L_1 \le t < h(x)\}$, occurs as a union of level sets in C_i.

If a t-level set is contained in the base X for some t, we call the level set a cut. The cuts in a flow column form a skyscraper of the base. Recall that $P = \{P_0, P_1\}$ is a partition of X according to the values of f. We let $L_2(\delta, \varepsilon)$ be an integer such that the subset without the property $E_r(T, L_2, P, \varepsilon/2)$ has measure less than $\delta/2$. The following Lemma is analogous to Lemma 2.

LEMMA 3. *Given any δ and ε, there exists a flow skyscraper such that the relative measure of the cuts in each column without the property $E_r(T, L_2, P, \varepsilon/2)$ is less than δ.*

COROLLARY 2. *There exists a subset $B \subset X$ such that the flow skyscraper built with B satisfies both Lemma 2 and Lemma 3.*

Proof. We denote by D the base of the flow skyscraper in Lemma 3. Consider the set $D[0, 1-\alpha) = \{(x,t) : x \in D,\ 0 \le t < 1-\alpha\}$ instead of A in the proof of Lemma 2 to find a subset B of D such that the flow skyscraper with B satisfies $(*)$. Clearly this skyscraper satisfies both Lemmas.

Let C_i denote the i^{th} column of a flow skyscraper and $\{\Lambda_{i,j}\}$ $(j = 1, 2, \ldots, m_i)$ denote the collection of cuts in the column C_i. Let $\{\Lambda'_{i,j}\}$ $(j = 1, 2, \ldots, m_i)$ be a collection of flat level sets in C_i. We let $X' = \bigcup_i \bigcup_j \Lambda'_{i,j}$. The following easy Lemma is basic to our construction.

LEMMA 4. *There exist a transformation T', a σ-algebra β' and a measure μ' on X' such that the flow can be built with the base (X', T', β', μ') which is isomorphic to (X, T, β, μ).*

Proof. We let $x' \in \Lambda'_{i,j}$. We denote by x the point in $\Lambda_{i,j}$ directly above or below x', according to the location of $\Lambda_{i,j}$ relative to $\Lambda'_{i,j}$ in the column. We call this x the corresponding point to x' and vice versa. We define T' on X' by defining T' on each of $\Lambda'_{i,j}$s. If $x' \in \Lambda'_{i,j}$ where $1 \le j < m_i$, then we define $T'(x')$ to be the point in $\Lambda'_{i,j+1}$ directly above x'. If $x' \in \Lambda'_{i,m_i}$, then we define $T'(x')$ to be the corresponding point to $T(x) \in \Lambda_{k,1}$, where $x \in \Lambda_{i,m_i}$ is the corresponding point to x'. Since there is an obvious α-algebra and a measure on each of $\Lambda'_{i,j}$s, there is an obvious σ-algebra β' and a measure μ' on X'. Clearly φ mapping each point in $\Lambda_{i,j}$ to its corresponding point in $\Lambda'_{i,j}$ for all i and j is an isomorphism between (X', T', β', μ') and (X, T, β, μ).

LEMMA 5. *Given δ, ε and η where $\delta < \varepsilon$ and $\delta < \eta/8 < 1/8$, there exists a positive integer K such that for any $k \geq K$, there exists an induced skyscraper denoted by $\Omega'(\subset \Omega)$ with the following properties:*

(1) *Each column C'_i in Ω' consists of level sets, $\{C'_{i,j}\}$, each of which satisfies*

  (1.1) *the heights of $C'_{i,j}$s are between $k(1-\alpha)$ and $k(1-\alpha)+2-\alpha$*

  (1.2) *$\{(x,t) : x \in B_{i,j},\ 0 \leq t < 1-\alpha\}$ has the property $E_r(S, k, Q, \varepsilon)$ where $B_{i,j}$ denotes the base of the level set $C'_{i,j}$*

  (1.3) *The first cut in each level set $C'_{i,j}$ has the property $E_r(T, k', P, \varepsilon)$ where $k' = [(1-\alpha)k]$*

(2) *$\nu(\Omega - \Omega') < \delta/2$.*

Proof. We let $L_1 = L_1(\delta^3/2, \varepsilon/2)$ and $L_2 = L_2(\delta^3/2, \varepsilon/2)$. Let $K = \max\{L_1, [L_2/1-\alpha]+1\}$. We may assume that K is sufficiently large so that $1/K < \delta^2$. We call the set R_0 without the property $E_r(S, K, Q^{\varepsilon/4}, \varepsilon/8)$ a bad set. We note that the measure of R_0 is less than $\delta^3/4$. We call any cut without the property $E_r(T, [K(1-\alpha)], P, \varepsilon/2)$ a bad cut. (Note that $[K(1-\alpha)] \geq L_2$.) The measure of all bad cuts is less thant $\delta^3/4$.

We fix $k \geq K$ and let $k' = [(1-\alpha)k]$. We build a flow skyscraper fo heights bigger than $(1-\alpha)k^3$ such that each column according to Q-names satisfies Lemma 2 and 3. That is, the relative measure of R_0 and of bad cuts in each column is less than $\delta^3/2$. Denote the base of the skyscraper by B. Let C_1 and B_1 denote the first column and its base respectively.

We consider the set $B_1[0, 1-\alpha) = \{(x,t) : x \in B_1,\ 0 \leq t < 1-\alpha\}$. If this set has the property $E_r(S, K, Q, \varepsilon/2)$ and the first cut contained in $B_1[0,1)$ has the property $E_r(T, [K(1-\alpha)], P, \varepsilon/2)$, then we proceed up to t_1 where $t_1 = \min\{t : (x,t) \in X \text{ and } t > (1-\alpha)k\}$. We call this level set $B_1[0, t_1)$ a good block. In this case we say that the block has the property $E_r(S, K, Q, \varepsilon/2)$ instead of saying that the bottom $B_1[0, 1 - \alpha)$-level set has the property. Also we say that the block has the property $E_r(T, [K(1 - \alpha)], P, \varepsilon/2)$ instead of saying that the first cut in the block has the property. Clearly a good block has the properties $E_r(S, k, Q, \varepsilon/2)$ and $E_r(T, k', P, \varepsilon/2)$.

If either $B_1[0, 1 - \alpha)$ has a subset without the property $E_r(S, K, Q, \varepsilon/2)$ or the first cut in $B_1[0, 1)$ is bad, then we remove the set $B_1[0, 1 - \alpha)$. In this case, we let $t_1 = 1 - \alpha$.

In either case, we consider the next level set $B_1[t_1, t_1 + (1 - \alpha))$. If this set has the property $E_r(S, K, Q, \varepsilon/2)$ and the first cut contained in $B_1[t_1, t_1 + 1)$ has the property $E_r(T, [K(1 - \alpha)], P, \varepsilon/2)$, then we proceed up to $t_2 = \min\{t : (x, t) \in X \text{ and } t > (1 - \alpha)k + t_1\}$. If the set does not satisfy both conditions, then we remove the set and consider the next level set of length $1 - \alpha$. We repeat this until we reach the t-level set such that we cannot build a good block any more, i.e., $h(x) - t < (1 - \alpha)k$. We remove the set $B_1[h(x) - r(1 - \alpha), h(x))$ where $r = [(h(x) - t)/(1 - \alpha)]$. That is, we remove a level set of length $r(1 - \alpha)$ from the top of the column. We place the level set $B_1[t, h(x) - r(1 - \alpha))$ on top of the last good block. This set is so small compared with a good block that we may ignore this "end effect" and assume that the last block has the property $E_r(S, K, Q, \varepsilon/2)$.

Now we compute how much of the skyscraper was removed in the column. There are three cases for a level set to be removed.

(i) A level set of length $1 - \alpha$ which has a subset without the property $E_r(S, K, Q, \varepsilon/2)$.
(ii) A level set of length $1 - \alpha$, $B_1[t, t + (1 - \alpha))$, if the first cut contained in $B[t, t + 1)$ is bad
(iii) A level set at the top of a column, $B_1[h(x) - r(1 - \alpha), h(x))$.

From Corollary 1, if there is a level set $B_1[t, t')$ which does not have the property $E_r(S, K, Q, \varepsilon/2)$, then $B_1(t - \varepsilon\eta/8, t' + \varepsilon\eta/8)$ does not have the property $E_r(S, K, Q^{\varepsilon/4}, \varepsilon/8)$. Hence, whenever we remove a level set of length $1 - \alpha$ because of (i), it contains a set of length at least $\varepsilon\eta/8$ without the property $E_r(S, K, Q^{\varepsilon/4}, \varepsilon/8)$. That is, it contains a level set of length $\varepsilon\eta/8$ which is contained in R_0. We let M_1 denote the number of level sets of length $1 - \alpha$ to be removed because of (i). Since the relative measure (relative length) of the bad set R_0 in each column is less than $\delta^3/2$ by our construction of the skyscraper, we have

$$M_1(\varepsilon\eta/8) < (\delta^3/2)h(x)$$
$$M_1 < 4\delta^3 h(x)/\varepsilon\eta < \delta h(x)/2$$

Hence the relative measure of the set to be removed in a column because of (i) is

$$M_1(1-\alpha) < \delta h(x)(1-\alpha)/2 < \delta h(x)/4.$$

We note that it is possible for a subset of the bad set R_0 to be contained in good blocks.

For each bad cut, there are at most $[1/(1-\alpha)]+1$ many level sets of length $1-\alpha$ to be removed. We let M_2 denote the number of level sets of length $1-\alpha$ to be removed because of (ii). We have

$$M_2 < ([1/(1-\alpha]+1)(\delta^3/2)([h(x)/\alpha]+1).$$

Recall that we assume $\alpha > 1/2$. Hence $[h(x)/\alpha]+1 < 2[h(x)]$ is the upper bound of the number of cuts in a column. Therefore the relative measure of the bad set removed in the case of (ii) is less than

$$\begin{aligned}
&\frac{([1/1-\alpha]+1)(1-\alpha)(\delta^3/2)([h(x)/\alpha]+1)}{h(x)} \\
&< \frac{([1/(1-\alpha)]+1)(1-\alpha)(\delta^3/2)2h(x)}{h(x)} \\
&< 2\delta^3 < \delta/8.
\end{aligned}$$

Also the relative measure of the set removed from the top in the case of (iii) is at most

$$k(1-\alpha)/k^3(1-\alpha) < 1/k^2 < \delta/8.$$

Hence the total measure of the set removed from the skyscraper is at most $\delta/2$.

We put all good blocks on top of each other and call the new column C_1'. We repeat this for each column. Hence we have a new flow skyscraper (induced flow skyscraper) consisting only of good blocks. Clearly this skyscraper satisfies (1.2), (1.3) and (2). We recall that to construct each good block except the last one we proceeded up to the height t_k so that $(x, t_k) \in X$. Hence the heights of these good blocks are between $k(1-\alpha)$ and $k(1-\alpha)+1$. The last good block in each column could have extra level set whose length is smaller than $(1-\alpha)$. Hence the new skyscraper satisfies (1.1). This completes the proof.

It should be noted that the set removed between two consecutive blocks is a level set whose length is a multiple of $(1-\alpha)$. We denote the flow induced on the good blocks by $(\Omega', S'_t, \mathcal{F}', \nu')$.

THEOREM 1. *The flow $(\Omega', S'_t, \mathcal{F}', \nu')$ can be represented as a $\{1, \alpha\}$-flow on a base, denoted by (X', T', β', μ'), which is isomorphic to (X, T, β, μ) where $X' \subset \Omega'$.*

Proof. Let ω_1 denote the integer so that $\omega_1(1-\alpha)$ is the sum of the heights of the set to be removed in the column C_1 to build Ω' in Lemma 5. We have

$$\begin{aligned}
\omega_1 &< M_1 + M_2 + k \\
\omega_2 &< 4([\frac{\delta^3 h(x)}{\varepsilon\eta}]) + k + ([\frac{1}{1-\alpha}] + 1)\delta^3 h(x) \\
&< \delta h(x)/2 + k + e\delta^3 h(x)
\end{aligned}$$

where $e = [1/(1-\alpha)] + 1$.

Recall that the column C_1 satisfies Lemma 2 and Lemma 3. Let n ($\gg L_1$) be the number of cuts in the column C_1. We let n_0 and n_1 denote the number of cuts which are contained in P_0 and P_1 respectively. (Recall that each cut is completely contained in P_0 or P_1.) We may assume that the column is long enough so that

$$n_1 > (\mu P_1 - \varepsilon)n \gg \omega_1 \tag{**}$$

We call a $[t_1, t_1+1)$-level set a 1-level set if the t_1-level set and the (t_1+1)-level set are cuts. We call a $[t_1, t_1+\alpha)$-level set a 0-level set if the t_1-level set and the $(t_1+\alpha)$-level set are cuts. To construct a new base (X', T', β', μ'), it is enough to redefine the cuts so that the number of cuts in the new column remains the same and the distance between two consecutive cuts is either 1 or α (see Lemma 4). We accomplish this by changing enough 1-level sets to 0-level sets. Hence before we remove a level set, denoted by $B_1[t, t + (1 - \alpha))$, we shift the cuts as follows. [See Figure 1]. To be consistent with the Figure, we will say right or left instead of up or down.

Case 1. The set to be removed does not contain a cut.

(i) It is contained in a 1-level set. We leave the cuts as they are and remove the set. This 1-level set has been changed to 0-level set.

(ii) It is contained in a 0-level set (recall $1 - \alpha < \alpha$). Let Λ denote the cut where this 0-level set begins. If there exists a 1-level set to its left, then we move all the cuts between the next cut to the right of Λ' and Λ inclusive to the left by $1 - \alpha$ where Λ' denotes the cut where the 1-level set begins. If there is no 1-level set to its left, then we let Λ' denote the cut where the first 1-level set, which is to the right of Λ, begins. We move all the cuts between the next cut to the right of Λ and Λ' inclusive to the right by $1 - \alpha$. Then we remove the set. We note that the 0-level set remains as a 0-level set.

Case 2. The set to be removed contains a cut Λ.

If there exists a cut Λ' to the left of Λ, where a 1-level set begins, then we move all the cuts between the next cut to the right of Λ' and Λ inclusive to the left by $1 - \alpha$. If there exists no 1-level set to the left of Λ, then we consider a cut Λ' to the right of Λ, where a 1-level set begins and we move all the cuts between Λ and Λ' inclusive to the right by $1 - \alpha$. Then we remove the level set.

We note that in either case, no cut is removed.

By property $(**)$, for each set of length $(1 - \alpha)$ to be removed, we can always find a 1-level set either to its left or to its right. We define T' on each of new cuts as in Lemma 4. We repeat this for all columns. If we let X' be the union of all new cuts with the obvious σ-algebra and the measure, then it is clear that the induced flow on Ω' is built under the heights 1 and α with the new base $(X', T', \mathcal{F}', \mu')$ which is isomorphic to $(X, T, \mathcal{F}, \mu)$.

REMARK 1. Let ω_i denote the number of removed sets of length $1 - \alpha$ from each column C_i. More generally, if the number of 1-level

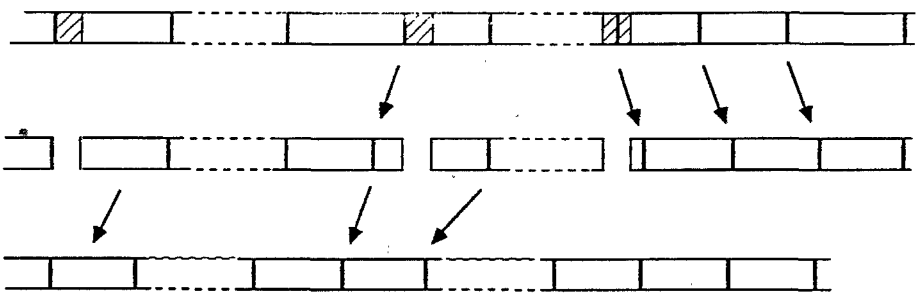

FIGURE 1

sets in each column is bigger than ω_i, then it is clear from the proof of Theorem 1 that the induced flow can be built under 1 and α with an isomorphic base.

When Δ denotes a level set, we denote by $[[\Delta]]$ the number of cuts in the level set. We note that $[[\Delta]]$ is always between $[\ell]$ and $[\ell/\alpha]+1(< 2[\ell])$ where ℓ is the length of the level set Δ.

LEMMA 6. *Given any small δ and $\varepsilon > 0$, we can construct good blocks in Lemma 5 so that any level set at the bottom or top of a good block, whose relative length in a good block is between δ and $1-\delta$, has the property $E(T,[[\Delta]],P,\varepsilon)$.*

Proof. We let $L_1 = (\delta^3/2, \varepsilon/2)$ and $L_2 = L_2(\delta^3/2, \varepsilon\delta/10)$, and $K = \max\{L_1, [L_2/(1-\alpha)]+1\}$. Choose $k \geq [K/\delta]+1$. We construct good blocks of length between $k(1-\alpha)$ and $k(1-\alpha)+(2-\alpha)$ as in Lemma 5. Each block $C_{i,j}$ has the property $E_r(S,K,Q,\varepsilon/2)$ and $E_r(T,[K(1-\alpha)],P,\varepsilon\delta/10)$, hence $E(S,k,Q,\varepsilon/2)$ and $E(T,[[C_{i,j}]],P,\varepsilon\delta/10)$. Let ℓ_0 and ℓ denote the length of a good block and its level set Δ respectively.

If the level set Δ is at the bottom of a good block, then clearly it has the property $E(T,[[\Delta]],P,\varepsilon)$ by the choice of K and k and the condition on ℓ.

Let Δ be at the top of a good block. We note that the rest of the level set $C_{i,j}\backslash\Delta$ in the block is a level set at the bottom. Since its length is bigger than $\delta k(> K)$, it has the property $E(T,[[C_{i,j}\backslash\Delta]],P,\varepsilon\delta/10)$.

If we assume that the set Δ does not have the property $E(T, [[\Delta]], P, \varepsilon)$ (that is, the error of the level set Δ is more than ε), then the error of the good block has to be at least

$$\begin{aligned}\frac{\varepsilon[\ell] - (\varepsilon\delta/10)(2[\ell_0 - \ell])}{2[\ell_0]} &\geq \frac{\varepsilon}{2}\left(\frac{\ell - 1}{\ell_0 + 1} - (\varepsilon\delta/10)(\frac{\ell_0 - \ell + 1}{\ell_0 - 1})\right) \\ &\geq (\varepsilon\delta/4 - \varepsilon\delta/10) > \varepsilon\delta/10.\end{aligned}$$

This is a contradiction.

LEMMA 7. *Given any small δ, there exist Ω' and X' in Theorem 1 such that $\nu(\Omega') > 1 - \delta(1-\alpha)/100$ and $\mu(X \cap X') > 1 - \delta$.*

Proof. Let $\xi = \delta^3(1-\alpha)/100$. Choose $K = \max\{L_1, [L_2/1-\alpha]+1\}$ where $L_1 = L_1(\xi, \varepsilon/2)$ and $L_2 = L_2(\xi, \varepsilon\delta/10)$. We fix $k > 10(K/\delta)$ and construct good blocks for this k as in Lemma 5. By the choice of K, hence of k, a good block has the property that any level set Δ of length ℓ between $\delta k(1-\alpha)/10$ and $(1-\delta)k(1-\alpha)/10$ has the property $E(T, [[\Delta]], P, \varepsilon)$. Hence a level set of length ℓ has at least $[\ell/2]$ 1-level sets. (We assume that $\varepsilon < (\mu(P_1) - 1/2)$). In each level set whose cuts are to be changed we will change one quarter of the 1-level sets to 0-level sets.

Let $\omega_{i,j}$ denote the number of the sets of length $1-\alpha$ to be removed between the j^{th} good block and the $j+1^{th}$ good block in the column C_i. We note that, for each level set of length $1-\alpha$ to be removed, we change at most one 1-level set to a 0-level set.

If $\omega_{i,j} < (1/4)(1/2)[\delta k(1-\alpha)/10]$, then it is enough to change the cuts in a level set whose relative length in a good block is at most $\delta/10$. If $\omega_{i,j}$ is bigger than $[(\delta/10)k(1-\alpha)]/8$ and $\omega_{i,j}(1-\alpha)$ is less than the length of a good block, then the cuts in a level set of length at most ℓ, where ℓ satisfies $\omega_{i,j} < (1/4)[\ell/2] < \omega_{i,j}+1$, have to be shifted. That is, ℓ is less than $8(\omega_{i,j}+1)$. If $\omega_{i,j}(1-\alpha)$ is larger than the length of a good block, then the length ℓ of a level set whose cuts are shifted is at most $8(\omega_{i,j}+1) + \delta k(1-\alpha)/10$. By the proof of Lemma 5 and the choice of ξ, the relative measure of the set to be removed in each column is less that $\xi' = \delta(1-\alpha)/100$. Hence we have

$$\sum_j \omega_{i,j}(1-\alpha) < \delta(1-\alpha)h(x)/100.$$

If we denote by κ the total length of those level sets whose cuts are shifted, then we have

$$\begin{aligned}\kappa &< \sum_j (8(\omega_{i,j}+1) + \delta k(1-\alpha)/10)\\ &= 8\sum_j(\omega_{i,j}+1) + (\delta/10)\sum_j k(1-\alpha)\\ &< 16\delta h(x)/100 + \delta h(x)/10 < (\delta/2)h(x).\end{aligned}$$

Since there are at most $[(1/\alpha)(\delta/2)h(x)] < [\delta h(x)]$ cuts in the union of level sets whose cuts are changed, while the number of cuts in the column is at least $[h(x)]$, the Lemma is clear.

REMARK 2. If a $[s, s')$-level set is once used to change 1-level sets to 0-level sets, then we do not use the level set again to change its 1-level sets to 0-level sets. Hence each good block still has at least $(1/2 - 1/8)k(1-\alpha)/2$ 1-level sets.

REMARK 3. We can choose δ small enough so that each new column has more 1-level sets than 0-level sets, after the shifts of cuts.

II.2 First step toward mixing

We let $Q = \{Q_{0,1}Q_{0,2}, Q_{1,1}, Q_{1,2}\}$ be a partition of the flow where

$$\begin{aligned}Q_{0,k} &= \{(x,t) : x \in P_0,\ (k-1)\alpha/2 \le t < k\ \alpha/2\} \text{ for } k = 1,2,\\ Q_{1,k} &= \{(x,t) : x \in P_1,\ (k-1)/2 \le t < k/2\} \text{ for } k = 1,2.\end{aligned}$$

Let η_1 be $\alpha/2$ which is the height of the partition Q. Let $\varepsilon_1, \varepsilon_2$ and δ_1 be given so that $\delta_1 < \varepsilon_2 < \varepsilon_1 < (\mu P_1 - 1/2)$ and $\delta_1 < \eta_1/8$. Let $e = [1/(1-\alpha)] + 1$ and $\xi_1 = \delta_1^3(1-\alpha)/100$. Choose $K_1 > \max\{L_1, [L_2/1-\alpha]+1\}$ where $L_1 = L_1(\xi_1, \varepsilon_1/2)$ and $L_2 = L_2(\xi_1, \varepsilon_1\delta_1/10)$.

We let L denote the maximum height of a flow skyscraper satisfying Lemma 2 and Lemma 3 for K_1 and $[K_1(1-\alpha)]$ respectively. We may assume that K_1 is large enough so that any level set Δ whose relative length in a good block is between $\delta/10$ and $(1-\delta)/10$ has the property $E(T, [[\Delta]], P, \varepsilon)$ as in Lemma 6. Let D be the base of the flow skyscraper. We choose positive integers a_1, b_1, c_1 and d_1 such that

(i) $3/b_1 < \varepsilon_1$

(ii) $16\dfrac{b_1}{a_1}\dfrac{K_1+q}{K_1}e < \delta_1$, where a_1 is a multiple of b_1

(iii) $1/c_1 < \delta_1/10$ $(< \varepsilon_2/10)$

(iv) $\dfrac{8ec_1(a_1+b_1)(K_1+q)}{d_1} < \delta_1/4$, where q is the integer satisfying $(q-1)(1-\alpha) < 2-\alpha < q(1-\alpha)$.

Construct a flow skyscraper of heights bigger than $2d_1(1-\alpha)$ with a base B which is a subset of D. We partition the skyscraper according to Q-names. Let us fix a column $C(1)$. We will remove level sets in several steps. Before we remove any level set, we always shift the cuts first to make sure that there remains the same number of cuts in the column.

Step 1. As in Lemma 5, we build good blocks of length between $K_1(1-\alpha)$ and $K_1(1-\alpha)+(2-\alpha)$. We call them 1-blocks. We convert the column $C(1)$ into a column $C(2)$ consisting of 1-blocks as in Lemma 5. The relative measure of the set to be removed is less than $\xi_1 = \delta_1(1-\alpha)/100$. (1.1)

To build the flow on the remaining set with a base isomorphic to (X, T, β, μ), we move the cuts as in Lemma 7 (before we remove the set) so that the relative measure of the set of the shifted cuts is less than δ_1. (1.2)

We note that there are at least $3[K_1(1-\alpha)]/8$ 1-level sets left in each 1-block. (See Remark 2)

Step 2. We will remove a set from the top whose length is a multiple of $(1-\alpha)$ and at most $(c_1(a_1+b_1)-1)(K_1+q)(1-\alpha)$ to form a new column $C(3)$ so that the number of 1-blocks in $C(3)$ is a multiple of $c_1(a_1+b_1)$.

We note that by (iv), the measure of the set to be removed is less than $\delta_1/4$. (2.1)

Let ℓ $(< (c_1(a_1+b_1)-1)(K_1+q)(1-\alpha))$ denote the length of the set to be removed from the top of the column $C(2)$. In each 1-block from the top of $C(3)$, we change $[K_1(1-\alpha)]/8$ many 1-level sets to 0-level sets and slide down all the cuts above them until we have a set of length $\ell+\alpha$ without any cuts on the top. Then we remove the level set of length ℓ from the top. Let M_1 denote the number of 1-blocks whose cuts have to be changed. We have

$$M_1[\frac{K_1(1-\alpha)}{8}](1-\alpha) < (c_1(a_1+b_1)-1)(K_1+q)(1-\alpha)$$

$$\begin{aligned} M_1 &< 8\frac{(c_1(a_1+b_1)-1)(K_1+q)(1-\alpha)}{[K_1(1-\alpha)](1-\alpha)} \\ &\leq 8(c_1(a_1+b_1)-1)(K_1+q)/[K_1(1-\alpha)] \\ &\leq 8(c_1(a_1+b_1)-1)\frac{K_1+q}{K_1}[1/(1-\alpha)]. \end{aligned}$$

The total number of 1-blocks whose cuts are changed is less than $8ec_1(a_1+b_1)(K_1+q)/K_1$. Since there are at least $d_1(1-\alpha)/((K_1+q)(1-\alpha))$ 1-blocks in the column, by (iv) the proportion of the number of these blocks is less than $\delta_1/4$. Recall that in each column the relative measure of cuts in a level set is at most twice the relative length of the level set. Hence the relative measure of the cuts which are changed at this step is less than $\delta_1/2$. (2.2)

We note that in each 1-block there are more than $[K_1(1-\alpha)]/4$ 1-level sets left.

Step 3. We call a_1 1-blocks followed by b_1 1-blocks a 1-pile. In this case we say that 1-pile consists of an a_1-pile and a b_1-pile. Now we form a skyscrapter $G(1)$ such that each column has c_1 1-piles by putting $c_1(a_1+b_1)$ consecutive 1-blocks in $C(3)$ next each other. We will convert all b_1-piles into "independent filters" except the top b_1-pile as follows. (See [5] for details.)

Let t_1 be the height of the first b_1-pile. Divide the first 1-pile into $m_1 = [t_1/(1-\alpha)]$ subpiles of equal width. We enumerate these subpiles by $V_{1,0}, V_{1,1}, \ldots, V_{1,m_1-1}$. From the top of each subpile $V_{1,k}$, we remove a level set of length $k(1-\alpha)$ for $k = 0, 1, \ldots, m_1-1$. We denote the remaining part of $V_{1,k}$ by $W_{1,k}$.

Each subpile gives rise to a subcolumn of the column. Let t_2 denote the height of the second b_1-pile. We change the b_1-pile in the next 1-pile followed by $W_{1,k}$ into independent filters as we did in the first b_1-pile for each $k = 0, \ldots, m_1-1$. Since each $W_{1,k}$ gives rise to m_2 subcolumns at this step, we have $m_1 m_2$ subcolumns where $m_2 = [t_2/1-\alpha]$. We repeat this for every b_1-pile one at a time except for the top b_1-pile. Furthermore, in every column of $G(1)$ we change all b_1-piles inot independent filters except the last one. We denote the remaining skyscraper by $G(2)$.

The measure of the set removed at this step in making filters is less than $(1/2)(b_1/a_1) < \delta_1/32$ by (ii). (3.1)

We compute the measure of the cuts which have t be shifted before we remove these sets. Let ρ_1 be the distance between the last cut in the b_1-pile and the top of the b_1-pile. We fix $V_{1,k}$. Each time we change a b_1-pile into independent filters, we first change enough 1-level sets to 0-level sets and move down the cuts above them so that there is a level set of length $k(1-\alpha)+\rho_1$ without cuts. And then we remove the level set of length $k(1-\alpha)$ from the top. Let M_2 denote the number of consecutive 1-blocks whose half of the remaining 1-level sets are to be changed to 0-level sets. Since there are at least $[K_1(1-\alpha)]/4$ 1-level sets, we have

$$M_2\frac{[K_1(1-\alpha)]}{8}(1-\alpha) < (K_1+q)(1-\alpha)b_1$$

$$M_2 < \frac{8[K_1+q)b_1}{[K_1(1-\alpha)]} < 8b_1\frac{K_1+q}{K_1}([1/(1-\alpha)]+1)$$

Since each 1-pile has (a_1+b_1) 1-blocks, by the choice of a_1 in (ii) we have $M_2/a_1 < \delta_1/2$. Hence the relative measure of the cuts which are changed at this step is less than δ_1. (3.2)

Repeat (1), (2) and (3) for each column of the skyscraper and denote the remaining skyscraper by G_1. Combining (1.1), (2.1) and (3.1) in all columns, we remove the set whose measure is less than δ_1. It is clear from (1.2), (2.2) and (3.2) that the total measure of the cuts that we have changed is less than $3\delta_1$.

REMARK 4. Note that the level set of length $m_i(1-\alpha)$ from the top of the i^{th} b_1-pile is used to construct independent filters. Since t_i is not necessarily a multiple of $1-\alpha$, there may be a level set at the bottom of the b_1-pile of length $t_i - m_i(1-\alpha)$ $(< 1-\alpha)$ left above the top 1-block of the i^{th} a_1-pile. We include this set in the top 1-block of the a_1-pile. Since this level set is so small compared with a 1-block, we may assume that this "end effect" is absorbed in our error $\varepsilon_1/2$. That is, the top 1-block of the a_1-pile still satisfies $E(S, [\tau(1-\alpha)], Q, \varepsilon_1/2)$ where τ is the height of the top 1-block.

REMARK 5. Each 1-block in G_1 has at least $[K_1(1-\alpha)]/8$ 1-level sets. We can assume that δ is small enough that there are more 1-level sets than 0-level sets in each column of G_1.

We denote this induced flow on G_1 by $(\Omega^{(1)}, S_t^{(1)}, \mathcal{F}^{(1)}, \nu^{(1)})$ and the new base by $(X^{(1)}, T^{(1)}, \beta^{(1)}, \mu^{(1)})$. The new base is a union of new cuts with the obvious transformation as in Lemma 5. Therefore, we have $\mu(X \cap X^{(1)}) > 1 - 3\delta_1$ and $\nu(\Omega \cap \Omega^{(1)}) > 1 - \delta_1$. Moreover the flow $((\Omega^{(1)}, S_t^{(1)}, \mathcal{F}^{(1)}, \nu^{(1)})$ is represented as a $\{1, \alpha\}$-flow with the base $(X^{(1)}, T^{(1)}, \beta^{(1)}, \mu^{(1)})$ which is isomorphic to (X, T, β, μ). Let $P^{(1)} = \{P_0^{(1)}, P_1^{(1)}\}$ be the partition of $X^{(1)}$ where $P_0^{(1)}$ is a union of cuts where 0-level sets start and $P_1^{(1)}$ is a union of cuts where 1-level sets start.

II.3 Induction Step

Recall that $P = P^1 \subset P^2 \cdots$ is an increasing sequence of finite partitions of X such that $V_1^\infty P^i$ generate the σ-algebta β. We let $P^n = \{P_1^n, P_2^n, \ldots, P_{v_n}^n\}$. We define $Q^n = \{Q_{1,1}^n, Q_{1,2}^n, \ldots, Q_{1,2^{n+1}}^n, Q_{2,1}^n, \ldots, Q_{2,2^{n+1}}^n, \ldots, Q_{v_n,1}^n, \ldots, Q_{v_n,2^{n+1}}^n\}$ to be a partition of $(\Omega, S_t, \mathcal{F}, \nu)$ where

$$Q_{i,j}^n = \{(x,t) : x \in P_i^n,\ \frac{j-1}{2^{n+1}}\alpha \le t < \frac{j}{2^{n+1}}\alpha\}$$
$$\text{if } P_i^n \subset P_0 \text{ for } j = 1, \ldots, 2^{n+1}.$$
$$Q_{i,j}^n = \{(x,t) : x \in P_i^n,\ \frac{j-1}{2^{n+1}} \le t < \frac{j}{2^{n+1}}\}$$
$$\text{if } P_i^n \subset P_1, \text{ for } j = 1, \ldots, 2^{n+1}.$$

It is clear that $Q^1 \subset Q^2 \subset Q^3 \subset \cdots$ is an increasing sequence of partitions of Ω and $V_i^\infty Q^i$ generates the σ-algebra of Ω. Let $Q_{i,j}^{*n} = Q_{i,j}^n \cap \Omega^{(n-1)}$ for $i = 1, \ldots, v_n$ and $j = 1, \ldots, 2^{n+1}$. Let Q^{*n} denote the partition $\{Q_{i,j}^{*n}\}$ of $\Omega^{(n-1)}$. We refine Q^{*n} so that the partition is rectangular. For notational convenience we will denote $S^{(n-1)}$ by S and the refinement of Q^{*n} by Q in this section.

We denote by H_{n-1} the maximum height of the flow skyscraper G_{n-1}. Let A_{n-1} be the base of G_{n-1} and $g_{n-1}(x)$ be the height of

G_{n-1} for each $x \in A_{n-1}$. Let the width of the smallest column of G_{n-1} be ζ_n. We subpartition G_{n-1} according to Q names. Let q_n denote the number of columns of G_{n-1} and $\{A_{n-1,i}\}$ denote the bases of columns. Let the height of Q be η_n. We choose a decreasing sequence $\{\varepsilon_n\}$ such that $100\sum_n \varepsilon_n < \mu(P_1) - 1/2$. We let $\gamma_n = \eta_n\zeta_n/[H_{n-1}/1-\alpha]$. We choose δ_n such that $\delta_n < \varepsilon_{n+1}$ and $\delta_n < \varepsilon_n\gamma_n/2^{n+1}$. Clearly we have $2\sum_{k\geq n}\delta_k < \varepsilon_n\gamma_n$ and we can assume $\delta_n < \eta_n/8$. Recall that $e = [1/1-\alpha]+1$ and q is the smallest integer satisfying $2-\alpha < q(1-\alpha)$. Let $\xi_n = \delta_n^3(1-\alpha)/100H_{n-1}$. Choose an integer K_n such that

(a) $K_n > \max\{L_1, \{L_2/1-\alpha]+1\}$ where $L_1 = L_1(\xi_n, \varepsilon_n/2)$ and $L_2 = L_2(\xi_n, \varepsilon_n\delta_n/10)$ with respect to the flow $((\Omega^{(n-1)}, S_t^{(n-1)}, Q, \mu^{(n-1)})$ and $(X^{(n-1)}, T^{(n-1)}, P^{(n-1)}, \nu^{(n-1)})$ respectively.

(b) $\dfrac{H_{n-1}+1-\alpha}{K_n(1-\alpha)} < \delta_n^2/2$

(c) K_n is sufficiently large that an n-block of length between $K_n(1-\alpha)$ and $K_n(1-\alpha)+H_{n-1}$ satisfies Lemma 6 for $\delta = \delta_n/10$ and $\varepsilon = \varepsilon_n$.

Choose positive integers a_n, b_n, c_n and d_n such that

(i) $3/b_n < \varepsilon_n$

(ii) $b_nc_n(K_n(1-\alpha)+H_{n-1}) < a_nH_{n-1}$

(iii) $16e(b_n/a_n)((K_n+q)(1-\alpha)+H_{n-1})/K_n(1-\alpha)) < \delta_n/2$, where a_n is a multiple of b_n

(iv) $1/c_n < \delta_n/10$ $(< \varepsilon_{n+1}/10)$

(v) $8ec_n(a_n+b_n)((K_n+q)(1-\alpha)+H_{n-1})/d_nH'_{n-1} < \delta_n/10$, where H'_{n-1} is the minimum height of G_{n-1}.

Consider the induced flow $S_t^{(n-1)}$ on G_{n-1}. We denote the induced transformation on A_{n-1} by U_{n-1}. We build a skyscraper of (A_{n-1}, U_{n-1}) whose heights are bigger than $2d_n$. The base of the tower is denoted by B_n. We build a flow skyscraper over the base B_n. By Corollary 2, we may assume that the flow skyscraper satisfies Lemma 2 and Lemma 3. Notice that the heights of the flow skyscraper are bigger than $2d_nH'_{n-1}$. We subpartition the skyscraper over B_n according to Q names and fix a column $C_{n,i}(1)$. We denote the base of the column by $B_{n,i}$. Let $h_{n,i}$ denote the height of the column. As in section II.2, we divide our work into several steps.

Step 1. If the set $B_{n,i}[0,1-\alpha)$ has the property $E_r(S, K_n, Q, \varepsilon_n/2)$ and the first cut (the first level set contained in $X^{(n-1)}$ in $B_{n,i}[0,1)$) has the property $E_r(T^{(n-1)}, [K_n(1-\alpha)], P^{(n-1)}, \varepsilon_n\delta_n/4)$, then we proceed up to the height of at least $K_n(1-\alpha)$ until the top of a subcolumn of G_{n-1} is reached. This can be done because we constructed the flow skyscraper over the base $B_n \subset A_{n-1}$. In this case we call the level set of length between $K_n(1-\alpha)$ and $K_n(1-\alpha)+H_{n-1}$ an n-block.

If the set $B_{n,i}[0,1-\alpha)$ does not have the property $E_r(S, K_n, Q, \varepsilon_n/2)$ or the first cut in $B_{n,i}[0,1)$ does not have the property $E_r(T^{(n-1)}, [K_n(1-\alpha)], P^{(n-1)}, \varepsilon_n\delta_n/4)$, then we will remove the set $B_{n,i}[0,\ell)$ where $\ell = [g_{n-1}(x)/(1-\alpha)](1-\alpha)$ for $x \in B_{n,i}$.

In either case, we consider the next level set of length $1-\alpha$, $B_{n,i}[t, t+1-\alpha)$, above an n-block or above $B_{n,i}[0,\ell)$. We proceed inductively, either forming an n-block or removing a set $B_{n,i}[t, t+\ell)$, where ℓ is the largest multiple of $1-\alpha$ so that $t+\ell$ is smaller than the top of the next subcolumn of G_{n-1} above the t-level set.

When we are left with a level set of length ℓ less than $K_n(1-\alpha)$ from the top, then we will remove the level set of length $[\ell/(1-\alpha)](1-\alpha)$ from the top. Besides n-blocks we may have a set of length less than $1-\alpha$ left in the column. We put this set on top of the last n-block. Hence each n-block except the top one is a concatenation of subcolumns of G_{n-1}, possibly having an additional set of length less than $1-\alpha$ at the bottom. The top n-block may have additional sets of length less than $1-\alpha$ at both ends. Denote the new column consisting of n-blocks by $C_{n,i}(2)$.

Each n-block has length between $K_n(1-\alpha)$ and $K_n(1-\alpha)+G_{n-1}+2(1-\alpha) < (K_n+q)(1-\alpha)+G_{n-1}$. As in Remark 4, we may assume that each of these n-blocks has the properties $E(S, [\tau/(1-\alpha)], Q, \varepsilon_n/2)$ and $E(T^{(n-1)}, N, P^{(n-1)}, \varepsilon_n\delta_n/4)$ where τ is the height of an n-block and N is the number of cuts in an n-block.

Recall that a set without the property $E(S, K_n, Q^{\varepsilon_n/4}, \varepsilon_n/8)$ is called a bad set. By the choice of L_2, the relative measure of the bad set in each column is less than ξ_n. Each time we remove a level set, it has a length at most $H_{n-1}+(1-\alpha)$. We compute the measure of the level sets removed from $C_{n,i}(1)$ in the construction of $C_{n,i}(2)$.

Let I_1 denote the number of level sets to be removed because each of them contains a subset without the property $E_r(S, K_n, Q, \varepsilon_n/2)$. Recall that each of these level sets has a bad level set whose length is at least $\varepsilon_n\eta_n/8$.

Recall that δ_n is less than $\eta_n/8$. Since each column satisfies Lemma 2, we have

$$I_1 \frac{\varepsilon_n \eta_n}{8} < \xi_n h_{n,i} < \frac{\delta_n^3(1-\alpha)h_{n,i}}{100H_{n-1}}$$
$$I_1 < \frac{8\delta_n^3(1-\alpha)h_{n,i}}{100\varepsilon_n\eta_n H_{n-1}} < \frac{\delta_n(1-\alpha)h_{n,i}}{100H_{n-1}}$$

Hence the measure of removed set is less than

$$I_1(H_{n-1}+(1-\alpha)) < \frac{\delta_n(1-\alpha)(H_{n-1}+1-\alpha)}{100H_{n-1}}h_{n,i} < \frac{\delta_n(1-\alpha)}{50}h_{n,i}$$

Let I_2 denote the number of removed level sets to form $C_{n,i}(2)$ because the first cut contained in $B_{n,i}[t,t+1)$ is a bad cut. Keep in mind that there are at most $2[h]$ cuts in a level set of length h. Since each column satisfies Lemma 3, we have

$$I_2 < 2\xi_n[h_{n,i}] < \frac{2\delta_n^3(1-\alpha)}{100H_{n-1}}[h_{n,i}].$$

The measure of the removed set is at most

$$I_2(H_{n-1}+(1-\alpha)) < \frac{\delta_n^3(1-\alpha)(H_{n-1}+1-\alpha)}{50H_{n-1}}h_{n,i} < \frac{\delta_n^3(1-\alpha)}{25}h_{n,i}.$$

Hence the relative measure of the union of level sets removed at this step is less than $\delta_n(1-\alpha)/25$.

As in the proof of Lemma 7, we will compute the measure of the cuts to be changed. In each n-block whose cuts are to be changed, we change every fourth 1-level set to a 0-level set. As in Remark 5, we may assume that by our choice of $\varepsilon_n' s$, hence of $\delta_n' s$, there are more 1-level sets than 0-level sets in each column of G_{n-1}. Let $\omega_{i,j}$ denote the number of removed sets of length at most $H_{n-1}+(1-\alpha)$ between j^{th} and $j+1^{th}$ good blocks. If

$$\omega_{i,j}\left([\frac{H_{n-1}}{1-\alpha}]+1\right) < \frac{1}{4}\frac{1}{2}\frac{\delta_n K_n(1-\alpha)}{10},$$

then the cuts in a level set whose relative length in a good block is at most $\delta_n/10$ have to be changed. If

$$\omega_{i,j}\left([\frac{H_{n-1}}{1-\alpha}]+1\right) > \frac{1}{4}\frac{1}{2}\frac{\delta_n K_n(1-\alpha)}{10} \quad \text{and}$$

$\omega_{i,j}(1-\alpha)$ is less than the minimum length of n-blocks, then the cuts in a level set of length at most ℓ have to be shifted where ℓ is the largest integer satisfying

$$\frac{1}{4}\frac{\ell}{2} < (\frac{H_{n-1}+1-\alpha}{1-\alpha})\,(\omega_{i,j}+1).$$

If $\omega_{i,j}(1-\alpha)$ is larger than the length of an n-block, then we have to change the cuts in a level set of length at most

$$\frac{\delta_n K_n(1-\alpha)}{10} + 8\frac{H_{n-1}+1-\alpha}{1-\alpha}(\omega_{i,j}+1).$$

Therefore the whole length of the level sets whose cuts are shifted in the column is at most

$$\sum(\frac{\delta_n K_n(1-\alpha)}{10} + \frac{8(H_{n-1}+1-\alpha)(\omega_{i,j}+1)}{1-\alpha})$$
$$< \frac{\delta_n}{10}\sum K_n(1-\alpha) + \frac{8}{1-\alpha}\sum(\omega_{i,j}+1)(H_{n-1}+1-\alpha).$$

By (b) of the condition on K_n, we may assume that

$$\sum(H_{n-1}+1-\alpha) < \delta_n(1-\alpha)h_{n,i}/100.$$

Since $\sum\omega_{i,j}(H_{n-1}+1-\alpha) < \frac{\delta_n(1-\alpha)h_{n,i}}{25}$, the whole length is

$$< \frac{\delta_n}{10}h_{n,i} + \frac{8}{1-\alpha}\left(\frac{\delta_n(1-\alpha)}{25} + \frac{\delta_n(1-\alpha)}{100}\right)h_{n,i}$$
$$< (\frac{\delta_n}{10} + \frac{\delta_n}{2})h_{n,i} < \delta_n h_{n,i}.$$

Therefore the measure of the cuts that are shifted is less than $2\delta_n h_{n,i}$.

Once a level set is used to change its 1-level sets to 0-level sets, then we do not use the set again to change its remaining 1-level sets to 0-level sets within this step. Hence we can safely assume that every n-block has at least $(3/8)[K_n(1-\alpha)]$ 1-level sets and every $(n-1)$-block in an n-block has at least $(3/32)[K_{n-1}(1-\alpha)]$ 1-level sets.

Step 2. We will remove a set from the top whose length is a multiple of $1-\alpha$ and at most $(c_n(a_n+b_n)-1)(K_n+[H_{n-1}/1-\alpha]+1)(1-\alpha)$ so that the number of n-blocks in each column is a multiple of $c_n(a_n+b_n)$. Since each column has height at least $2d_nH'_{n-1}$, by (v) the measure of the set to be removed at this time is less than $\delta_n/160$.

Again, before we remove the set, we change every fourth one of the remaining 1-level sets to 0-level sets and shift the cuts down so that the level set to be removed does not contain any cuts. Let M_1 denote the number of n-blocks whose cuts are to be changed.

$$M_1\frac{1}{4}\frac{3[K_n(1-\alpha)]}{8} < (c_n(a_n+b_n)-1)(K_n+[H_{n-1}/1-\alpha]+1)$$
$$M_1 < \frac{32}{3}(c_n(a_n+b_n)-1)\left(\frac{K_n+[H_{n-1}/1-\alpha]+1}{[K_n(1-\alpha)]}\right)$$
$$< (32/3)(c_n(a_n+b_n)-1)2e < (64/3)ec_n(a_n+b_n).$$

According to (v) the proportion of the number of n-blocks with the cuts to be changed is less than $\delta_n/6$, so that the relative measure of the changed cuts is less than $\delta_n/3$.

Step 3.i. We now form a new skyscraper $G_{n,1}$ such that each column in $G_{n,1}$ has $c_n(a_n+b_n)$ n-blocks. We call a_n n-blocks followed by b_n n-blocks an n-pile. We say that an n-pile consists of an a_n-pile and a b_n-pile.

We convert into independent filters all the top sub-b_{n-1}-piles in n-blocks which are contained in a_n-piles as in section II.2. We remove the set of measure at most $(1/2)1/c_{n-1} < \delta_{n-1}/20$ by (iv). We denote the remaining skyscraper by $G_{n,2}$.

Let N be the number of sub $(n-1)$-blocks whose cuts are to be changed. Since each $(n-1)$-block still has at least $3[K_{n-1}(1-\alpha)]/32 >$

$[K_{n-1}(1-\alpha)]/16$ 1-level sets, we have

$$N\frac{[K_{n-1}(1-\alpha)]}{16}(1-\alpha) < b_{n-1}(K_{n-1}(1-\alpha)+H_{n-2})$$
$$N < 16b_{n-1}\frac{K_{n-1}(1-\alpha)+H_{n-2}}{[K_{n-1}(1-\alpha)]}\frac{1}{1-\alpha} < 32b_{n-1}\frac{1}{1-\alpha}.$$

The relative length of the level set whose cuts are to be changed is less than

$$\frac{32b_{n-1}(1/(1-\alpha))}{c_{n-1}(a_{n-1}+b_{n-1})} < \frac{1}{c_{n-1}} < \frac{\delta_{n-1}}{10}$$

by (iii) and (iv). Hence the relative measure of the cuts to be changed is less than $\delta_{n-1}/5$. Since δ_{n-1} is sufficiently small, every n-block still has at least $[K_n(1-\alpha)/4]$ 1-level sets after this step.

After the formation of these filters, each n-block in an a_n-pile yields several columns in $G_{n,2}$ which we still call n-blocks. Since the original n-blocks in an a_n-pile have the property $E_r(S, K_n, Q, \varepsilon_n/2)$, the n-blocks in an a_n-pile have the property $E_r(S, K_n, Q, 3\varepsilon_n/4)$. (Recall that δ_{n-1} is less than ε_n).

Step 3.ii. In each column of $G_{n,2}$ we convert the b_n-piles into independent filters as in section II.2 except the top b_n-pile. We denote this skyscraper by G_n. The measure of the removed set to form G_n from $G_{n,2}$ is less than $\delta_n/32$ by (iii).

As in Step 3 of II.2, we let M_2 denote the number of consecutive n-blocks in which half of the remaining 1-level sets are changed to 0-level sets. We have

$$M_2\frac{[K_n(1-\alpha)]}{8}(1-\alpha) < (K_n(1-\alpha)+H_{n-1})b_n$$
$$M_2 < \frac{8b_n(K_n(1-\alpha)+H_{n-1})}{[K_n(1-\alpha)](1-\alpha)} < 16b_n e.$$

The relative measure of the level set whose cuts are changed is less than $\delta_n/2$ by (iii). Hence the relative measure of the cuts which are shifted is less than δ_n.

Combining (1), (2), (3.i) and (3.ii), it is easy to see that the total measure of the set removed is less than $\delta_n + \delta_{n-1}$ and the measure of the cuts changed is less than $2\delta_n + \delta_n/3 + \delta_{n-1}/5 + \delta_n < 4\delta_n + \delta_{n-1}/5$.

We note that each n-block in G_n has at least $[K_n(1-\alpha)]/8$ 1-level sets. By our choice of $\delta_n' s$, there are more 1-level sets than 0-level sets in each column of G_n.

This completes the construction of the flow $(\Omega^{(n)}, S_t^{(n)}, \mathcal{F}^{(n)}, \nu^{(n)})$ with the base $(X^{(n)}, T^{(n)}, \beta^{(n)}, \mu^{(n)})$ which is a union of new cuts and is isomorphic to (X, T, β, μ). By our choice of $\delta_n' s$, it is clear that the sequences $\{(\Omega^{(n)}, S_t^{(n)}, \mathcal{F}^{(n)}, \nu^{(n)})\}$ and $\{(X^{(n)}, T^{(n)}, \beta^{(n)}, \mu^{(n)})\}$ converge to the limits denoted by $(\tilde{\Omega}, \tilde{S}_t, \tilde{\mathcal{F}}, \tilde{\nu})$ and $(\tilde{X}, \tilde{T}, \tilde{\beta}, \tilde{\mu})$ respectively. $(\tilde{X}, \tilde{T}, \tilde{\beta}, \tilde{\mu})$ is isomorphic to (X, T, β, μ) by the usual argument.

References

1. W. Ambrose,, *Representation of ergodic flows*, Ann. of Math. **42** (1942), 723–739.
2. A. del Junco and K. Park,, *An example of a measure-preserving flow with minimal self-joinings*, J. D'analyse Mathematique **42** (1982/83), 199–209.
3. A. del Junco, A. Fieldsteel, D. Rudolph, α-equivalence : a refinement of Kakutani equivalence, preprint..
4. A del Junco, A. Fieldsteel, K. Park,, *residual properties of α-flows*, in preparation.
5. N. Friedman and D. Ornstein,, *Ergodic transformations induce mixing transformations*, Advances in Mathematics **10** (1973), 147–163.
6. A. Katok,, *Interval exchange transformations Israel J. Math.*.
7. D. Ornstein,, *Ergodic Theory, Randomness and Dynamic Systems*, Yale University Press, New Haven (1974).
8. K. Park,, *Three Bernoulli factors that generate an ergodic flow*, Springer Lecture Note, the Special Year Dynamical System.
9. ——— ,, *Nice dense subsets for ergodic flows and Bernoulli flows*, Pacific J. of Math. **119** (1985), 181–190.
10. ——— ,, *Two non-isomorphic flows with the same very weak Bernoulli partition on a base*, J. of Math. Analysis and Appl. **113** (1986), 255–265.
11. ——— ,, *A short proof of α-equivalence*, preprint.
12. D. Rudolph,, *A two-valued step coding for ergodic flows*, Mathematische Zeitschrift, (1976), 201–220.
13. ——— ,, *Restricted orbit equivalence*, Memoirs of Amer. Math. Soc. **54** (1985).
14. P. Shields,, *The Theory of Bernoulli shifts*, Chicago University Press, Chicago (1973).

Department of Mathematics
Ajou University
Suwon 441-749, Korea