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THE APPROXIMATE EIGENVALUES OF AN OPERATOR MATRIX

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In [5, 6] R. E. Harte determined the spectrum of an "operator matrix" by computing the joint spectrum for certain systems of elements in a tensor product of normed algebras and applying a spectral mapping theorem in several variables. In this note we derive a spectral mapping theorem for the joint approximate eigenvalues in a tensor product of normed algebras and then determine the approximate eigenvalues of an operator matrix.

If $a = (a_1, a_2, \cdots, a_n) \in A^n$ is an n-tuple of elements in a normed algebra A then we write

$$\tilde{\sigma}_A^{\ell}(a) = \Big\{\lambda \in \mathbf{C}^n : 1 \notin \text{ cl } \sum_{j=1}^{n} A(a_j - \lambda_j)\Big\}$$

and

$$\tilde{\sigma}_A^{r}(a) = \Big\{\lambda \in \mathbf{C}^n : 1 \notin \text{ cl } \sum_{j=1}^{n} (a_j - \lambda_j)A\Big\}$$

for the almost left spectrum and the almost right spectrum, respectively, of $a \in A^n$ with respect to A. We can make similar extension to n-tuples of approximate eigenvalues ([2], [5], [6]): we shall write

$$\tilde{\tau}_A^{\ell}(a) = \Big\{\lambda \in \mathbf{C}^n : \inf_{\|x\| \geq 1} \sum_{j=1}^{n} \|(a_j - \lambda_j)x\| = 0\Big\}$$

and

$$\tilde{\tau}_A^{r}(a) = \Big\{\lambda \in \mathbf{C}^n : \inf_{\|x\| \geq 1} \sum_{j=1}^{n} \|x(a_j - \lambda_j)\| = 0\Big\}$$

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for the *left approximate eigenvalues* and the *right approximate eigenvalues*, respectively, of a with respect to A. We have the obvious inclusions:

$$\tilde{\tau}^{\ell}_A(a) \subseteq \tilde{\sigma}^{\ell}_A(a) \qquad \text{and} \qquad \tilde{\tau}^{r}_A(a) \subseteq \tilde{\sigma}^{r}_A(a).$$

If $a \in A^n$ then for each $\omega \in \{\tilde{\sigma}^l_A, \tilde{\sigma}^r_A, \tilde{\tau}^l_A, \tilde{\tau}^r_A\}, \omega(a)$ is a compact subset of $\mathbf{C}^n$. The "spectral mapping theorem" for $\omega \in \{\tilde{\sigma}^l_A, \tilde{\sigma}^r_A, \tilde{\tau}^l_A, \tilde{\tau}^r_A\}$ is the assertion that if $a \in A^n$ is commutative and $f : \mathbf{C}^n \to \mathbf{C}^m$ is an m-tuple of polynomials in n variables then ([1], [2], [5], [6])

$$\omega f(a) = f\omega(a). \tag{0.1}$$

If $\omega \in \{\tilde{\sigma}^l_A, \tilde{\sigma}^r_A, \tilde{\tau}^l_A, \tilde{\tau}^r_A\}$ then ω has a "subprojective" property: that is, if $a \in A^n$ and $b \in A^m$ are arbitrary then

$$\omega(a,b) \subseteq \omega(a) \times \omega(b).$$

If ω is a subprojective system of mappings from A^n into subsets of $\mathbf{C}^n$ we shall write

$$\omega_{b=\mu}(a) = \{\lambda \in \mathbf{C}^n : (\lambda, \mu) \in \omega(a,b)\}.$$

It was known ([4] Theorem 2.3; [6] Theorem 11.3.5) that if $a \in A^n$ is commutative and commutes with $b \in A^m$ and if $f : \mathbf{C}^{n+m} \to \mathbf{C}^p$ is a p-tuple of polynomials in $n+m$ variables, then for each $\omega \in \{\tilde{\sigma}^l_A, \tilde{\sigma}^r_A, \tilde{\tau}^l_A, \tilde{\tau}^r_A\}$, there is equality

$$\omega f(a,b) = \bigcup_{\lambda \in \omega(a)} \omega_{a=\lambda} f(\lambda, b). \tag{0.2}$$

If A and B are complex normed algebras then we shall denote by $A \otimes B$ the completion of the algebraic "tensor product" $A \otimes_{\mathbf{C}} B$ with respect to some *uniform crossnorm* ([3]) which is compatible with the multiplication $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$. The space $A^{\dagger} \otimes B^{\dagger}$ can be naturally mapped into $(A \otimes B)^{\dagger}$: sometimes we have

$$A^{\dagger} \otimes B^{\dagger} \cong (A \otimes B)^{\dagger} \tag{0.3}$$

An obvious example for (0.3) is when A and B are both Frechet spaces and B is a nuclear space if a topological vector space ([10] Proposition 50.7) and hence, in particular, B is finite dimensional if a normed space.

Our first observation is elementary:

LEMMA 1. *If $a = (a_1, a_2, \cdots, a_n) \in A^n$ for a normed algebra A and if $A^\sim$ is the completion of A then*

$$\tilde{\tau}^l_A(a) = \tilde{\tau}^l_{A^\sim}(a) \qquad \text{and} \qquad \tilde{\tau}^r_A(a) = \tilde{\tau}^r_{A^\sim}(a). \tag{1.1}$$

Proof. If $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is not in $\tilde{\tau}^l_A(a)$ then there exists a positive constant $\epsilon > 0$ for which

$$\epsilon\|b\| \le \sum_{j=1}^n \|(a_j - \lambda_j)b\| \qquad \text{for all } b \in A.$$

Since for each $c \in A^\sim$, there exists a sequence (c_m) in A such that $c_m \to c$, it follows that

$$\epsilon\|c\| = \lim_{m\to\infty} \epsilon\|c_m\| \le \lim_{m\to\infty} \sum_{j=1}^n \|(a_j - \lambda_j)c_m\| = \sum_{j=1}^n \|(a_j - \lambda_j)c\|,$$

which says that λ is not in $\tilde{\tau}^l_{A^\sim}(a)$. This gives inclusion one way in the first equality of (1.1). The reverse inclusion is evident. Exactly similar argument gives the second equality of (1.1).

For tensor product of normed algebras, we have:

THEOREM 2. *If A and B are normed algebras and if $A \otimes B$ is a uniform tensor product of A and B satisfying $A^\dagger \otimes B^\dagger \cong (A \otimes B)^\dagger$, then for arbitrary $a \in A^n$ and $b \in B^m$ there is equality*

$$\tilde{\tau}^l_{A\otimes B}(a \otimes 1, 1 \otimes b) = \tilde{\tau}^l_A(a) \times \tilde{\tau}^l_B(b) \tag{2.1}$$

and

$$\tilde{\tau}^r_{A\otimes B}(a \otimes 1, 1 \otimes b) = \tilde{\tau}^r_A(a) \times \tilde{\tau}^r_B(b). \tag{2.2}$$

Proof. If $(\lambda, \mu) \in \mathbf{C}^{n+m}$ is not in $\tilde{\tau}^l_A(a) \times \tilde{\tau}^l_B(a)$ then there exists $\epsilon > 0$ for which either $\epsilon\|x\| \le \sum_{j=1}^n \|(a_j - \lambda_j)x\|$ for all $x \in A$ or $\epsilon\|y\| \le \sum_{k=1}^m \|(b_k - \mu_k)y\|$ for all $y \in B$. Now suppose, for each sequence $(\sum_{i=1}^{n_r} x_i^{(r)} \otimes y_i^{(r)})_{r\in\mathbf{N}}$ in $A \otimes B$,

$$\|\big((a_j - \lambda_j) \otimes 1\big) \sum_{i=1}^{n_r} x_i^{(r)} \otimes y_i^{(r)}\| \xrightarrow{r} 0 \qquad \text{for each } j$$

and

$$\Big\|(1\otimes(b_k-\mu_k))\sum_{i=1}^{n_r} x_i^{(r)}\otimes y_i^{(r)}\Big\| \xrightarrow{r} 0 \qquad \text{for each } k,$$

so that

$$\Big\|\sum_{i=1}^{n_r}(a_j-\lambda_j)x_i^{(r)}\otimes y_i^{(r)}\Big\| \xrightarrow{r} 0 \quad \text{and} \quad \Big\|\sum_{i=1}^{n_r} x_i^{(r)}\otimes(b_k-\mu_k)y_i^{(r)}\Big\| \xrightarrow{r} 0.$$

By the assumption that the tensor product is uniform, the linear functional

$$\phi\otimes\psi : \sum_{i=1}^{n} a_i\otimes b_i \longmapsto \sum_{i=1}^{n}\phi(a_i)\psi(b_i) \quad \text{for each } \phi\in A^\dagger \text{ and } \psi\in B^\dagger$$

is bounded, and extends to the product $A\otimes B$. We thus have, for each $\phi\in A^\dagger$ and $\psi\in B^\dagger$,

$$\sum_{i=1}^{n_r}\phi\big((a_j-\lambda_j)x_i^{(r)}\big)\psi(y_i^{(r)}) \xrightarrow{r} 0 \ \text{ and } \ \sum_{i=1}^{n_r}\phi(x_i^{(r)})\psi\big((b_k-\mu_k)y_i^{(r)}\big) \xrightarrow{r} 0,$$

so that

$$\phi\Big\{(a_j-\lambda_j)\sum_{i=1}^{n_r} x_i^{(r)}\psi(y_i^{(r)})\Big\} \xrightarrow{r} 0 \,\text{and}\, \psi\Big\{(b_k-\mu_k)\sum_{i=1}^{n_r} y_i^{(r)}\phi(x_i^{(r)})\Big\} \xrightarrow{r} 0,$$

and hence

$$(a_j-\lambda_j)\sum_{i=1}^{n_r} x_i^{(r)}\psi(y_i^{(r)}) \xrightarrow{r} 0 \quad \text{for each } j$$

and

$$(b_k-\mu_k)\sum_{i=1}^{n_r} y_i^{(r)}\phi(x_i^{(r)}) \xrightarrow{r} 0 \quad \text{for each } k.$$

Thus our assumption gives

$$\text{either } \sum_{i=1}^{n_r} x_i^{(r)}\psi(y_i^{(r)}) \xrightarrow{r} 0 \quad \text{or} \quad \sum_{i=1}^{n_r} y_i^{(r)}\phi(x_i^{(r)}) \xrightarrow{r} 0 :$$

Each case gives

$$\sum_{i=1}^{n_r} \phi(x_i^{(r)})\psi(y_i^{(r)}) \xrightarrow{r} 0 \quad \text{for each } \phi \in A^\dagger \text{ and } \psi \in B^\dagger,$$

and hence

$$\phi \otimes \psi \left(\sum_{i=1}^{n_r} x_i^{(r)} \otimes y_i^{(r)} \right) \xrightarrow{r} 0 \quad \text{for each } \phi \in A^\dagger \text{ and } \psi \in B^\dagger.$$

From the assumption that $A^\dagger \otimes B^\dagger \cong (A \otimes B)^\dagger$, we can conclude

$$\sum_{i=1}^{n_r} x_i^{(r)} \otimes y_i^{(r)} \xrightarrow{r} 0,$$

which, by (1.1), says that (λ, μ) is not in $\tilde{\tau}^l_{A\otimes B}(a \otimes 1, 1 \otimes b)$ because elements of the form $\sum_{i=1}^k a_i \otimes b_i$ form a dense subspace. This gives inclusion one way in (2,1). The reverse inclusion was noticed by Ichinose [8, Corollary 3.8]: Suppose $\lambda \in \tilde{\tau}^l_A(a)$ and $\mu \in \tilde{\tau}^l_B(b)$, so that there are sequences (x_r) and (y_r) in A and B for which $\|x_r\| = 1 = \|y_r\|$ and $\|(a_j - \lambda_j)x_r\| + \|(b_k - \mu_k)y_r\| \xrightarrow{r} 0$ for each j and k: but by the crossnorm property for $A \otimes B$

$$\|x_r \otimes y_r\| = 1$$

and

$$\big\|((a_j - \lambda_j) \otimes 1)(x_r \otimes y_r)\big\| + \big\|(1 \otimes (b_k - \mu_k))(x_r \otimes y_r)\big\| \xrightarrow{r} 0,$$

which says that $(\lambda, \mu) \in \tilde{\tau}^l_{A\otimes B}(a \otimes 1, 1 \otimes b)$. This proves (2.1). The argument for (2.2) is similar.

Theorem 2 gives us a spectral mapping theorem:

THEOREM 3. *Suppose that A and B are normed algebras and that $A \otimes B$ is a uniform tensor product of A and B satisfying $A^\dagger \otimes B^\dagger \cong (A \otimes B)^\dagger$. If $a \in A^n$ is commutative and commutes with $b \in B^m$ and if $f : \mathbf{C}^{n+m} \to \mathbf{C}^p$ is a p-tuple of polynomials in $n+m$ variables then*

$$\tilde{\tau}^l_{A\otimes B} f(a \otimes 1, 1 \otimes b) = \tilde{\tau}^l_B f(\tilde{\tau}^l_A(a), b) \tag{3.1}$$

and

$$\tilde{\tau}^r_{A\otimes B} f(a \otimes 1, 1 \otimes b) = \tilde{\tau}^r_B f(\tilde{\tau}^r_A(a), b). \tag{3.2}$$

Proof. From (0.2) and (2.1) we have

$$\begin{aligned}
&\tilde{\tau}^l_{A\otimes B} f(a \otimes 1, 1 \otimes b) \\
&= \bigcup_{\lambda \in \tilde{\tau}^l(a\otimes 1)} \tilde{\tau}^l_{a\otimes 1 = \lambda(1\otimes 1)} f(\lambda(1 \otimes 1), 1 \otimes b) \\
&= \bigcup_{\lambda \in \tilde{\tau}^l_A(a)} \tilde{\tau}^l_{a\otimes 1 = \lambda \otimes 1} f(1 \otimes \lambda), 1 \otimes b) \\
&= \bigcup_{\lambda \in \tilde{\tau}^l_A(a)} \{\mu \in \mathbf{C}^m : (\lambda, \mu) \in \tilde{\tau}^l(a \otimes 1, f(1 \otimes \lambda, 1 \otimes b))\} \\
&= \bigcup_{\lambda \in \tilde{\tau}^l_A(a)} \{\mu \in \mathbf{C}^m : (\lambda, \mu) \in \tilde{\tau}^l_A(a) \times \tilde{\tau}^l_B f(\lambda, b)\} \\
&= \{\mu \in \mathbf{C}^m : \mu \in \tilde{\tau}^l_B f(\lambda, b) \text{ for some } \lambda \in \tilde{\tau}^l_A(a)\} \\
&= \tilde{\tau}^l_B f(\tilde{\tau}^l_A(a), b),
\end{aligned}$$

which proves (3.1) and similarly (3.2).

Theorem 3 gives an expression for the approximate eigenvalues of an "operator matrix". If $\mathbf{C}_{nn}$ is the algebra of $n \times n$ complex matrices we shall write

$$A_{nn} = A \otimes C_{nn} \cong A^{n^2}$$

for the algebra of $n \times n$ matrices over the normed algebra A : All the uniform crossnorms give the same Cartesian product topology. If $x = (x_{ij}) \in A_{nn}$ is a commutative matrix (i.e., $x_{ij}x_{i'j'} = x_{i'j'}x_{ij}$

for each i, j, i', j') then we can define "determinant" exactly as in the numerical case ([7]):

$$\det(x)$$
$$= \sum \{\mathrm{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} : \sigma \in \mathrm{perm}(1, 2, \cdots, n)\} \in A$$

together with the "cofacter" or "adjugate" matrix

$$\mathrm{adj}(x) = (\tilde{x}_{ij}) \in A_{nn},$$

where $(-1)^{i+j}\tilde{x}_{ij}$ is the determinant of the submatrix of x obtained by deleting the row and column containing the entry x_{ij}. Evidently, exactly as in the numerical case

$$\mathrm{adj}(x)x = \det(x)I = x\mathrm{adj}(x),$$

where $I \in A_{nn}$ is the identity matrix. If $y = (y_{ij}) \in A_{nn}$ is another commuting matrix, whose entries commute with those of x, then we also see

$$\det(xy) = \det(x)\det(y) \in A.$$

It was known ([5], [6], [7], [9]) that if $x = (a_{ij}) \in A_{nn}$ is an $n \times n$ commutative matrix over the normed algebra A then

$$\tilde{\sigma}_{A_{nn}}(x) = \{\mu \in \mathbf{C} : 0 \in \tilde{\sigma}_A \det(x - \mu)\}. \tag{3.3}$$

For the approximate eigenvalues, we have an analog of (3.3):

THEOREM 4. *If $x = (a_{ij}) \in A_{nn}$ is an $n \times n$ matrix over the normed algebra A with a commuting sequence of entries $a = (a_{11}, a_{12}, \cdots, a_{1n}, \cdots, a_{nn}) \in A^{n^2}$ then*

$$\tilde{\tau}^l_{A_{nn}}(x) = \{\mu \in \mathbf{C} : 0 \in \tilde{\tau}^l_A det(x - \mu)\} \tag{4.1}$$

and

$$\tilde{\tau}^r_{A_{nn}}(x) = \{\mu \in \mathbf{C} : 0 \in \tilde{\tau}^r_A det(x - \mu)\}. \tag{4.2}$$

Proof. Writing $B = \mathbf{C}_{nn}$ let $b = (b_{11}, b_{12}, \cdots, b_{1n}, \cdots, b_{nn}) \in B^{n^2}$ be the canonical basis for the vector space B. Thus $x = (a_{ij}) \in A_{nn}$ can be written as

$$x = g(a, b) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \otimes b_{ij} \in A \otimes B = A_{nn}.$$

Since evidently $A^\dagger \otimes B^\dagger \cong (A \otimes B)^\dagger$, we have, by Theorem 3,

$$\begin{aligned} \tilde{\tau}^l_{A_{nn}}(x) = \tilde{\tau}^l_{A_{nn}} g(a,b) &= \tilde{\tau}^l_{A \otimes B} g(a \otimes 1, 1 \otimes b) \\ &= \tilde{\tau}^l_B g(\tilde{\tau}^l_A(a), b) \\ &= \bigcup_{\lambda \in \tilde{\tau}^l_A(a)} \tilde{\tau}^l_B \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} b_{ij} \Big). \end{aligned} \tag{4.3}$$

Since, for each $\lambda \in \mathbf{C}^n, g(\lambda, b)$ is an $n \times n$ complex matrix, it follows from the determinant theory in the numerical case that

$$\tilde{\tau}^l_B g(\lambda, b) = \{\mu \in \mathbf{C} : \det\ (g(\lambda, b) - \mu) = 0\}. \tag{4.4}$$

If we now apply (0,1) with the polynomial $f = \det(g(z, b) - \mu)$ then since $f(a) = \det(x - \mu)$ we have

$$0 \in \tilde{\tau}^l_A \det(x - \mu) \Longleftrightarrow \det(g(\lambda, b) - \mu) = 0 \text{ for some } \lambda \in \tilde{\tau}^l_A(a). \tag{4.5}$$

From (4.3), (4.4) and (4.5) we have

$$\begin{aligned} &\tilde{\tau}^l_{A_{nn}}(x) \\ =&\tilde{\tau}^l_B g(\tilde{\tau}^l_A(a), b) \\ =&\{\mu \in \mathbf{C} : \det(g(\lambda, b) - \mu) = 0 \text{ for some } \lambda \in \tilde{\tau}^l_A(a)\} \\ =&\{\mu \in \mathbf{C} : 0 \in \tilde{\tau}^l_A \det(x - \mu)\}, \end{aligned}$$

which gives (4.1) and similarly (4.2).

We believe that Theorem 4 can also be deduced from an extended version of [7, Theorem 2.2; 9, Lemma 1.1].

References

1. M. D. Choi and C. Davis, *The spectral mapping theorem for joint approximate point spectrum*, Bull. Amer. Math. Soc. **80** (1974), 317-321.
2. R. E. Curto, *Applications of several complex variables to multiparmeter spectral theory, Surveys of recent results in operator theory, vol II(J. B. Conway and B.B.Morrel eds.)*, Pitman Lec. Notes in Math. Ser. **192** (1989), 25-90 Longman Sc. and Techn., London.
3. J. Gil de Lamadrid, *Uniform crossnorms and tensor products of Banach algebras*, Duke Math. J. **32** (1965), 359-368.
4. R. E. Harte, *Spectral mapping theorem for quasicommuting systems*, Proc. Roy. Irish Acad. **73A** (1973), 7-18.
5. ______, *Spectral mapping theorems on a tensor product*, Bull. Amer. Math. Soc. **79** (1973), 367-372.
6. ______, *Invertibility and singularity for bounded linear operators*, Dekker, New York, 1988.
7. ______, *Invertibility and singularity of operator matrices*, Proc. Roy. Irish Acad. **88(A)** (1988), 103-118.
8. T. Ichinose, *Operational calculus for tensor products of linear operators in Banach spaces*, Hokkaido Math. J. **4 No. 2** (1975), 306-334.
9. N. Y. Krupnik, *Banach algebras with symbols and singular integral operators*, Birkhauser, Boston 1987.
10. F. Treves, *Topological vector spaces, distributions and kernels*, Academic Press, New York, 1967.

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