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AN ERROR ANALYSIS OF THE DISCRETE GALERKIN SCHEME FOR NONLINEAR INTEGRAL EQUATIONS

YOUNG-HEE KIM AND MAN-SUK SONG

1. Introduction

We employ the Galerkin method to solve the nonlinear Urysohn integral equation

$$x(t) = f(t) + \int_{\mathcal{D}} k(t, s, x(s))ds \quad (t \in \bar{\mathcal{D}}), \tag{1.1}$$

where $\mathcal{D}$ is a bounded domain in R^d, the function f and k are known and x is the solution to be determined. We assume that $\mathcal{D}$ has a locally Lipschitz boundary ([1, p. 67]). We can rewrite (1.1) in operator notation as

$$x = f + Kx.$$

We consider (1.1) as an operator equation on $L_\infty(\mathcal{D})$ and assume that K is defined on the closure $\bar{\Omega}$ of a bounded open set $\Omega \subset L_\infty(\mathcal{D})$. Throughout our analysis we put the following assumptions on (1.1).

- A1: $f \in L_\infty(\mathcal{D})$;
- A2: K is a completely continuous operator in $L_\infty(\mathcal{D})$, which has a fixed point x_0;
- A3: K is Fréchet differentiable at the solution x_0 of (1.1) and 1 is not an eigenvalue of the Fréchet derivative $L = K'(x_0)$;
- A4: For a given $x_0 \in \Omega$ and $\epsilon > 0$, assume that $K(x)$ is twice differentiable and $K''(x)$ is bounded on $B(x_0, \epsilon) \subset \Omega$.

The discrete forms of the Galerkin method and the iterated Galerkin method arise when the required integrals appeared in the two methods are calculated by means of the numerical integration.

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In the Galerkin method, we seek the approximation of the form

$$x_n = \sum_{i=1}^{n} a_i u_i,$$

as a solution of the equation (1.1) where the $\{u_i\}_{i=1}^{n}$ consist of given basis functions for a certain n-dimensional space $\mathcal{U}_n$ contained in $R(\mathcal{D})$. Here $R(\mathcal{D})$ is the space of bounded functions defined on $\mathcal{D}$ which are continuous almost everywhere and $\mathcal{U}_n$ will be taken to be of finite-element character(see Section 2 for further details).

The coefficients $\mathbf{a}_n = [a_1, \ldots, a_n]^T$ in x_n are obtained from the Galerkin equations

$$G_n \mathbf{a}_n - B_n(\mathbf{a}_n) = \mathbf{w}_n, \tag{1.2}$$

where G_n is the $n \times n$ Gram matrix having (i,j)th element $\langle u_i, u_j \rangle$, $B_n(\mathbf{a}_n)$ is the $n \times 1$ vector having i–th element $\langle u_i,\ K(\sum_{j=1}^{n} a_j u_j) \rangle$, $\mathbf{w}_n = [\langle u_1, f\rangle, \ldots, \langle u_n, f\rangle]^T$ and $\langle \cdot, \cdot \rangle$ denotes the usual L_2 inner product over $\mathcal{D}$. Here we use a L_∞ setting so that we obtain our error estimates in the L_∞ norm. Clearly, our assumptions about the basis functions and K ensure that the elements of G_n, B_n and $\mathbf{w}_n$ in (1.2) are well defined.

Assuming that such x_n exists, the iterated Galerkin solution x'_n is then given by

$$x'_n = f + Kx_n = f + K(\sum_{i=1}^{n} a_i u_i). \tag{1.3}$$

In (1.2), we see that the practical implementation of the Galerkin method requires the calculation of the integrals $\langle u_i,\ K(\sum_{j=1}^{n} a_j u_j) \rangle$ $(1 \leq i \leq n)$ in B_n and the integrals $\langle u_i,\ f \rangle$ $(1 \leq i \leq n)$ in $\mathbf{w}_n$, while calculating x'_n involves the integrals $K(\sum_{i=1}^{n} a_i u_i)$ in (1.3). To calculate these integrals, numerical integration is often used. Thus the actual equations to be solved are

$$G_n \widetilde{\mathbf{a}}_n - \widetilde{B}_n(\widetilde{\mathbf{a}}_n) = \widetilde{\mathbf{w}}_n \tag{1.4}$$

and the discrete Galerkin solution $\widetilde{x}_n$ is given by

$$\widetilde{x}_n = \sum_{i=1}^{n} \widetilde{a}_i u_i.$$

Here $\widetilde{\mathbf{a}}_n = [\widetilde{a_1}, \ldots, \widetilde{a}_n]^T$ is the numerical solution of the nonlinear algebraic system (1.4), which is obtained from an iteration method ([13]). $\widetilde{B}_n(\widetilde{\mathbf{a}}_n)$ and $\widetilde{\mathbf{w}}_n$ are the approximations of $B_n(\mathbf{a}_n)$ and $\mathbf{w}_n$ respectively when the numerical integrations are used to solve the integrals. When we approximate the integrals $K(\sum_{i=1}^{n} \widetilde{\mathbf{a}}_i u_i)$ by $\widetilde{K}(\sum_{i=1}^{n} \widetilde{\mathbf{a}}_i u_i)$, the discrete iterated Galerkin solution is given by

$$\widetilde{x}'_n = f + \widetilde{K}\widetilde{x}_n = f + \widetilde{K}(\sum_{i=1}^{n} \widetilde{a}_i u_i). \tag{1.5}$$

In 1987, Joe [7] gave an analysis of the discrete Galerkin scheme (that is, the discrete Galerkin and discrete iterated Galerkin methods) under suitable assumptions on the quadrature errors for the Fredholm integral equation of the second kind. In this paper, we generalize his results for the discrete Galerkin scheme for the linear equations into the nonlinear case.

In Section 2, we give some necessary background materials including estimates of the orders of convergence for the Galerkin scheme. We also review the prolongation and restriction operators with some of their properties, which are used to give our error analysis. The analysis of the discrete Galerkin scheme is given in Section 3, and some numerical examples are given in Section 4.

2. Preliminaries

For a non-negative integer m and $1 \leq p \leq \infty$, we equip the Sobolev space $W_p^m(\mathcal{D})$ with the usual norm $\|\cdot\|_{m,p,\mathcal{D}}$ ([1], p. 44). For convenience, we assume that all functions are real-valued.

For later use, we need to define P_n, the (unique) L_2 orthogonal projection onto the space $\mathcal{U}_n$. For all $\phi_n \in \mathcal{U}_n$, the operator P_n satisfies

$$\langle P_n g, \phi_n \rangle = \langle g, \phi_n \rangle \quad \text{for } g \in L_\infty(\mathcal{D}). \tag{2.1}$$

As mentioned in the Introduction, the Galerkin approximation x_n belongs to a finite-dimensional space $\mathcal{U}_n \subset R(\mathcal{D})$, which is of finite-element character. For our purpose, we do not give any detailed description of $\mathcal{U}_n$ is not required, but we shall assume that the space has typical properties of piecewise polynomials of degree $\leq r-1$, with r a positive integer. Specially, we shall assume the followings : first, $\mathcal{D}$ is partitioned into disjoint subregions of maximum (Euclidean) diameter $h \equiv h_n$, where h is related to n by $n \leq ch^{-\lambda}$ with $\lambda \geq 1$ (usually $\lambda = d$); secondly, the error of best L_p approximation by elements of $\mathcal{U}_n$ satisfies

$$(2.2) \qquad \inf_{\phi_n \in \mathcal{U}_n} \|g - \phi_n\|_{p,\mathcal{D}} \leq ch^{m^*} \|g\|_{m,p,\mathcal{D}} \quad \text{for } g \in W_p^m(\mathcal{D}),$$

where $m^* = \min\{m, r\}$ with c independent of p, n and g; and thirdly, P_n is a uniformly bounded operator on $L_p(\mathcal{D})$ ($1 \leq p \leq \infty$) so that

$$(2.3) \qquad \|P_n\|_{L_p} \leq c_1 < \infty$$

with c_1 independent of n and p. In this paper, $c, c_i (1 \leq i \leq 5)$ denote generic constants and will be independent of n.

For a given $\mathcal{U}_n$, we choose the basis functions $\{u_i\}_{i=1}^n$ of $\mathcal{U}_n$ satisfying some assumptions. Accordingly, we assume that the $\{u_i\}_{i=1}^n$ are such that the inverse Gram matrix G_n^{-1} satisfies

$$(2.4) \qquad \|G_n^{-1}\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |(G_n^{-1})_{ij}| \leq ch^{-\gamma}$$

for some $\gamma \geq 0$. Here $(G_n^{-1})_{ij}$ $(1 \leq i, j \leq n)$ denotes the (i,j)th element of G_n^{-1}. We also assume that the basis functions satisfy

$$(2.5) \qquad \| \sum_{i=1}^n |u_i| \ \|_{\infty,\mathcal{D}} \leq c$$

for some constant c. Since $\mathcal{U}_n \subset R(\mathcal{D}) \subset L_\infty(\mathcal{D})$ and $\mathcal{U}_n$ is of finite-element character, it is reasonable to give the assumption (2.5) to the basis functions having local support.

Now we give the following results concerning the properties of the Galerkin solution x_n and the iterated Galerkin solution x_n' ([2], [9]). We know that assumptions A2–A4 imply that x_0 is an isolated fixed point of K, of nonzero index ([9]).

THEOREM 2.1. *Suppose that* A1–A4, (2.2) *and* (2.3) *hold. Then for sufficiently large* n,

(i) x_n *and* x'_n *exist uniquely within some sufficiently small neighborhood of* x_0 *in* $\mathcal{U}_n$ *and* $L_\infty(\mathcal{D})$, *respectively,*

(ii) $\|x_0 - x_n\|_{\infty,\mathcal{D}} \le c_1 \|x_0 - P_n x_0\|_{\infty,\mathcal{D}}$,

(iii) $\|x_0 - x'_n\|_{\infty,\mathcal{D}} \le c_2 \|Kx_0 - KP_n x_0\|_{\infty,\mathcal{D}}$,

for some constants c_1 *and* c_2.

Proof. The results follow from Atkinson and Potra [3] and Atkinson [2].

In order to give quantitative estimates of the orders of convergence x_n and x'_n, we shall make some smooth assumptions on the solution x_0 and (the derivative of) the kernel k. We define l_t by $l_t(s) = k_u(t,s,x_0(s))$ for $t, s \in \bar{\mathcal{D}}$, where $k_u(t,s,u) = \frac{\partial}{\partial u} k(t,s,u)$.

THEOREM 2.2. *Suppose that the hypotheses of Theorem 2.1 hold; let* $x_0 \in W^l_\infty(\mathcal{D})$ *with* $l \ge 1$; *and let* $l_t \in W^m_1(\mathcal{D})$ $(t \in \bar{\mathcal{D}})$ *with* $m \ge 1$, *where* $\sup_{t'} \|l_{t'}\|_{m,1,\mathcal{D}} < \infty$. *Then*

(i) $\|x_0 - x_n\|_{\infty,\mathcal{D}} \le ch^{l^*} \|x_0\|_{l,\infty,\mathcal{D}}$,

(ii) $\|x_0 - x'_n\|_{\infty,\mathcal{D}} \le ch^{\phi} \|x_0\|_{l,\infty,\mathcal{D}}$,

where $\phi = \min\{m^* + l^*, 2l^*\}$ *with* $m^* = \min\{m, r\}$ *and* $l^* = \min\{l, r\}$.

Proof. In virtue of Theorem 2.1, it suffices to show

$$\|(I - P_n)x_0\|_{\infty,\mathcal{D}} \le ch^{l^*} \|x_0\|_{l,\infty,\mathcal{D}}, \tag{2.6}$$

for the proof of part (i). From the definition of P_n in (2.1), it is clear that $P_n \phi_n = \phi_n$ for all $\phi_n \in \mathcal{U}_n$ and hence

$$\|(I-P_n)x_0\|_{\infty,\mathcal{D}} = \|(I-P_n)(x_0-\phi_n)\|_{\infty,\mathcal{D}} \le (1+\|P_n\|_{L_\infty})\|x_0-\phi_n\|_{\infty,\mathcal{D}}.$$

Then the result (2.6) follows from (2.3) and (2.2) with an appropriate choice of ϕ_n.

To prove part (ii), we first note that (2.1) gives $\langle \psi_n, (I - P_n)x_0 \rangle = 0$ for any $\psi_n \in \mathcal{U}_n$. From the definition of L, we obtain

$$K(P_n x_0) - K(x_0) = L(P_n x_0 - x_0) + t_n, \tag{2.7}$$

with $t_n = o(\|P_n x_0 - x_0\|)$. The first term of the right side is bounded as following :

$$\begin{aligned}\|L(P_n x_0 - x_0)\|_{\infty,\mathcal{D}} &\le \sup_t |\langle l_t, (I - P_n)x_0\rangle| \\ &= \sup_t |\langle l_t - \psi_n, (I - P_n)x_0\rangle| \\ &\le \sup_t \|l_t - \psi_n\|_{1,\mathcal{D}} \|(I - P_n)x_0\|_{\infty,\mathcal{D}} \\ &\le ch^{m^*+l^*}\|x_0\|_{l,\infty,\mathcal{D}}.\end{aligned}$$

The last inequality follows from (2.2) with an appropriate choice of ψ_n and (2.6). Hence the result follows from (2.7) and Theorem 2.1 (iii).

REMARK 1. Let p and q satisfy $1/p + 1/q = 1$ with $1 \le p, q \le \infty$. By the general Hölder's inequality, we can show that Theorem 2.2(ii) is also true when $x_0 \in W_p^l(\mathcal{D})$ and $l_t \in W_q^m(\mathcal{D})$ with $\sup_{t'} \|l_{t'}\|_{m,q,\mathcal{D}} < \infty$.

Now we discuss the prolongation and restriction operators that are used for the analysis of the discrete Galerkin scheme in the next section. These operators serve as a link between $L_\infty(\mathcal{D})$, on which (1.1) is defined, and Euclidean space R^n in which the equations for the approximate solution are solved. Our treatment depends on that of Joe [7].

For $\mathbf{b}_n = [b_1, \ldots, b_n]^T \in R^n$, the *prolongation operator* $q_n : R^n \to \mathcal{U}_n$ is defined by

$$q_n \mathbf{b}_n = \sum_{i=1}^{n} b_i u_i.$$

By using the definition of the intermediate operator $s_n : L_\infty(\mathcal{D}) \to R^n$ given by

$$s_n g = [\langle u_1, g\rangle, \ldots, \langle u_n, g\rangle]^T \quad \text{for } g \in L_\infty(\mathcal{D}),$$

we define the *restriction operator* $r_n : L_\infty(\mathcal{D}) \to R^n$ by $r_n g = G_n^{-1} s_n g$.

Here we choose the norm in R^n to be

$$\|\mathbf{b}_n\|_{R^n} = \|q_n \mathbf{b}_n\|_{\infty,\mathcal{D}}$$

for $\mathbf{b}_n \in R^n$. From (2.5), we have a relation of the norm in R^n with the usual ∞–norm for R^n, $\|\mathbf{b}_n\|_\infty = \max_{1\le i\le n} |b_i|$ by

$$\|\mathbf{b}_n\|_{R^n} \le c_1 \|\mathbf{b}_n\|_\infty \le c_2 \|G_n^{-1}\|_\infty^{\frac{1}{2}} \|\mathbf{b}_n\|_{R^n}. \tag{2.8}$$

The second inequality follows from (3.2) in Joe [7]. Now we give the relationship between the induced matrix norm in R^n and the usual maximum row–sum norm for matrices.

LEMMA 2.3. *For any* $n \times n$ *matrix* A_n,

$$\|A_n\|_{R^n} \leq ch^{-\frac{1}{2}\gamma}\|A_n\|_\infty.$$

Proof. See Joe [7].

Now we summarize useful properties of q_n and r_n in the following lemma.

LEMMA 2.4.

(i) $r_n q_n = I_n$, *the identity in* R^n,

(ii) $q_n r_n = P_n$,

(iii) $\|q_n\| = 1$,

(iv) $\|r_n\| \leq c$, *for some constant* c.

Proof. Part (i) follows trivially from the definition of r_n and q_n with the orthonormality of $\{u_i\}$. We can find the proofs of the other part in Joe [7].

3. Discrete Galerkin scheme

In this section, we give the orders of convergence of the discrete Galerkin and discrete iterated Galerkin solutions. For our purposes, it is convenient to introduce the error operators E_1, E_2, E_3 and E_4. Suppose $\phi_n, \psi_n, \varphi_n \in \mathcal{U}_n \subset R(\mathcal{D})$ and $t, s \in \bar{\mathcal{D}}$. Then E_1, E_2, E_3 and E_4 are defined as follows:

$E_1(\phi_n, g) =$ quadrature error in evaluating the integral

$$\int_{\mathcal{D}} g(s)\phi_n(s)ds \quad \text{for } g \in R(\mathcal{D}).$$

$E_2(\phi_n, K\psi_n) =$ quadrature error in evaluating the integral

$$\int_{\mathcal{D}}\int_{\mathcal{D}} k(t, s, \psi_n(s))\phi_n(t)dsdt.$$

$E_3(\phi_n, K'(\varphi_n)\psi_n)$ = quadrature error in evaluating the integral

$$\int_{\mathcal{D}}\int_{\mathcal{D}} k_u(t,s,\varphi_n(s))\psi_n(s)\phi_n(t)dsdt.$$

$E_4(K\psi_n(t))$ = quadrature error at $t \in \bar{\mathcal{D}}$ in evaluating the integral

$$\int_{\mathcal{D}} k(t,s,\psi_n(s))ds.$$

The definition of E_2 includes the situation in which $K\psi_n$ is evaluated exactly and E_4 is only for the quadrature error at $t \in \bar{\mathcal{D}}$ in evaluating $K\psi_n$. To ensure that point evaluations of f (and hence $E_1(\phi_n, f)$) are well defined, we replace assumption A1 by

$$\text{A1}' : f \in R(\mathcal{D}).$$

From the definition of x_n, we can rewrite the Galerkin equations in (1.2) as

$$\langle u_i,\, x_n\rangle - \langle u_i,\, Kx_n\rangle \,=\, \langle u_i,\, f\rangle \quad (1 \le i \le n).$$

With the prolongation and restriction operators as defined in the previous section, we see from the above equation that the Galerkin equations in (1.2) may be written as

$$\mathbf{a}_n - r_n K(q_n \mathbf{a}_n) = r_n f. \tag{3.1}$$

Then the equations for the discrete Galerkin method given by (1.4) may be written as

$$\widetilde{\mathbf{a}}_n - r_n \widetilde{K}(q_n \widetilde{\mathbf{a}}_n) = f_n, \tag{3.2}$$

where $r_n\widetilde{K}(q_n\widetilde{\mathbf{a}}_n) = G_n^{-1}\widetilde{B}_n(\widetilde{\mathbf{a}}_n) \simeq r_nK(q_n\mathbf{a}_n) = G_n^{-1}B_n(\mathbf{a}_n)$ and $f_n = G_n^{-1}\widetilde{\mathbf{w}}_n \simeq r_nf = G_n^{-1}\mathbf{w}_n$.

We first obtain the following result for the discrete Galerkin method. Recall that $\gamma \ge 0$ is defined by $\|G_n^{-1}\|_\infty \le ch^{-\gamma}$.

THEOREM 3.1. *Suppose that* A1′, A2–A4, (2.2), (2.3) *and* (2.4) *hold; let* $x_0 \in W^l_\infty(\mathcal{D})$ *with* $l \geq 1$; *and let the quadrature errors be such that*

(a) $|E_1(u_i, f)| \leq ch^{\delta+\gamma}$ $(1 \leq i \leq n)$, $\delta > 0$,

(b) $|E_2(u_i, Kx_n)| \leq ch^{\xi+\gamma}$ $(1 \leq i \leq n)$, $\xi > 0$,

(c) $|E_3(u_i, K'(x_n)u_j)| \leq ch^{\mu+\frac{3}{2}\gamma}$ $(1 \leq i, j \leq n)$,
where $\mu > \lambda \geq 1$ *and* λ *satisfies* $n \leq ch^{-\lambda}$.

Then

$$\|x_0 - \widetilde{x}_n\|_{\infty,\mathcal{D}} \leq ch^\theta,$$

where $\theta = \min\{l, r, \delta, \xi\}$.

Proof. From (3.1) and (3.2), we obtain

$$\begin{aligned}\mathbf{a}_n - \widetilde{\mathbf{a}}_n =& r_n f - f_n + \{r_n K(q_n\mathbf{a}_n) - r_n\widetilde{K}(q_n\mathbf{a}_n)\} \\ &+ r_n\widetilde{K}'(q_n\mathbf{a}_n)(q_n\mathbf{a}_n - q_n\widetilde{\mathbf{a}}_n) + t_n,\end{aligned}$$

where $t_n = o(\|q_n\mathbf{a}_n - q_n\widetilde{\mathbf{a}}_n\|)$. From the definition of x_n, we can rewrite the above equation as

$$\begin{aligned}(I_n - r_n\widetilde{K}'(x_n)q_n)&(\mathbf{a}_n - \widetilde{\mathbf{a}}_n) \\ &= r_n f - f_n + \{r_n K(q_n\mathbf{a}_n) - r_n\widetilde{K}(q_n\mathbf{a}_n)\} + t_n.\end{aligned} \tag{3.3}$$

The left side of (3.3) is rewritten as $(I_n - G_n^{-1}\widetilde{H}_n)(\mathbf{a}_n - \widetilde{\mathbf{a}}_n)$, where $\widetilde{H}_n$ is the $n \times n$ matrix having (i,j)th element $\langle u_i, \widetilde{K}'(x_n)u_j\rangle$. Here we denote H_n the matrix having (i,j)th element $\langle u_i, K'(x_n)u_j\rangle$. From Lemma 2.3 and (2.4), we obtain

$$\begin{aligned}\|r_n K'(x_n)q_n - r_n\widetilde{K}'(x_n)q_n\|_{R^n} &= \|G_n^{-1}(H_n - \widetilde{H}_n)\|_{R^n} \\ &\leq ch^{-\frac{3}{2}\gamma}\|H_n - \widetilde{H}_n\|_\infty.\end{aligned}$$

It follows from the assumption (c) that

$$\|H_n - \widetilde{H}_n\|_\infty \leq nch^{\mu+\frac{3}{2}\gamma} \leq c_1 h^{\frac{3}{2}\gamma+\mu-\lambda},$$

and hence $\|r_n K'(x_n)q_n - r_n\widetilde{K}'(x_n)q_n\|_{R^n} \to 0$ as $n \to \infty$. From Lemma 2.4, $\|q_n\| = 1$ and $\|r_n\| \leq c$. Then the application of Theorem 3.2 in

Thomas [12] shows that for sufficiently large n, $(I_n - r_n\widetilde{K}'(x_n)q_n)^{-1}$ exists and is uniformly bounded. Hence from (3.2) and Theorem 2.1, we see that $\widetilde{\mathbf{a}}_n$ (and thus $\widetilde{x}_n$) exists uniquely.

From Theorem 2.2 (i), we have

$$\|x_0 - x_n\|_{\infty,\mathcal{D}} \leq ch^{\min\{l,r\}} \leq ch^\theta$$

and now we estimate the bound of $\|x_n - \widetilde{x}_n\|_{\infty,\mathcal{D}}$.

In terms of the prolongation operator q_n, we have

$$\|x_n - \widetilde{x}_n\|_{\infty,\mathcal{D}} = \|q_n\mathbf{a}_n - q_n\widetilde{\mathbf{a}}_n\|_{\infty,\mathcal{D}} = \|\mathbf{a}_n - \widetilde{\mathbf{a}}_n\|_{R^n}.$$

Since $\|(I_n - r_n\widetilde{K}'(x_n)q_n)^{-1}\|_{R^n}$ is uniformly bounded by c_2 (say), we obtain from (3.3) the fact that

$$\begin{aligned} \|x_n-\widetilde{x}_n\|_{\infty,\mathcal{D}} &= \|\mathbf{a}_n - \widetilde{\mathbf{a}}_n\|_{R^n} \\ &\leq c_2\{\|r_nf - f_n\|_{R^n} + \|r_nK(q_n\mathbf{a}_n) - r_n\widetilde{K}(q_n\mathbf{a}_n)\|_{R^n} \\ &\quad + c_3\|\mathbf{a}_n - \widetilde{\mathbf{a}}_n\|^2_{R^n}\}. \end{aligned} \tag{3.4}$$

We easily have from (2.8), (2.4) and assumption (a) of the theorem the fact that

$$\|r_nf - f_n\|_{R^n} \leq ch^\delta \leq ch^\theta.$$

For the second term of the right–hand side of (3.4), we use (2.8) to obtain

$$\begin{aligned} \|r_nK(q_n\mathbf{a}_n) - r_n\widetilde{K}(q_n\mathbf{a}_n)\|_{R^n} &\leq c_4\|r_nK(q_n\mathbf{a}_n) - r_n\widetilde{K}(q_n\mathbf{a}_n)\|_\infty \\ &\leq c_4\|G_n^{-1}\|_\infty\|B_n(\mathbf{a}_n) - \widetilde{B}_n(\mathbf{a}_n)\|_\infty \\ &\leq c_4h^{-\gamma}c_5h^{\xi+\gamma} \leq ch^\theta. \end{aligned}$$

Here the third inequality follows from (2.4) and assumption (b). This completes the proof.

REMARK 2. For consistency of the discrete Galerkin method, it is clear that we require $\delta, \xi \geq \min\{l, r\}$.

Now we let L^* be the adjoint of $L = K'(x_0)$ with respect to the inner product $\langle\cdot,\cdot\rangle$. Then we define the resolvent kernel $g_t(s)(t, s \in \bar{\mathcal{D}})$ by the unique function satisfying

$$(I - L^*)g_t = l_t \qquad (t \in \bar{\mathcal{D}}), \tag{3.5}$$

where l_t is given in Section 2. Let $l(t,s) \equiv l_t(s)$. Obviously, L is compact in $L_\infty(\mathcal{D})$ by assumption A3. Under the assumption on $l(t,s)$ given below in Theorem 3.2, it can be shown that L^* is compact in $L_1(\mathcal{D})$ and is a bounded operator from $L_1(\mathcal{D})$ to $W_1^m(\mathcal{D})$ ([10]). Hence we conclude that $g_t \in W_1^m(\mathcal{D})$ with $\sup_{t'} \|g_{t'}\|_{m,1,\mathcal{D}} < \infty$.

THEOREM 3.2. *Suppose that* A1′, A2–A4, (2.2), (2.3) *and* (2.4) *hold; let* $x_0 \in W_\infty^l(\mathcal{D})$ (*with* $l \geq 1$); *let* $\sup_t \|D_t^\alpha D_s^\beta l(t,\cdot)\|_{\infty,\mathcal{D}} < \infty$ *for* $0 \leq |\alpha| \leq z$ (*with* $0 < z \leq r$) *and* $0 \leq |\beta| \leq m$ (*with* $m \geq 1$); *and let the quadrature errors be such that*

(a) *condition* (c) *of Theorem 3.1 holds,*
(b) $\sup_t |E_4(K\widetilde{x}_n(t))| \leq ch^\rho, \rho > 0$,
(c) $\|x_0 - \widetilde{x}_n\|_{\infty,\mathcal{D}} \leq ch^\zeta, \ \zeta > 0$,
(d) $\sup_t |E_1(P_n g_t, f)| \leq ch^\eta, \eta > 0$,
(e) $\sup_t |E_2(P_n g_t, K\widetilde{x}_n)| \leq ch^\tau, \tau > 0$.

Then

$$\|x_0 - \widetilde{x}'_n\|_{\infty,\mathcal{D}} \leq ch^\sigma,$$

where $\sigma = \min\{\phi, \rho, \eta, 2\zeta, m^* + z + \zeta\}$ *with* $m^* = \min\{m, r\}$, $l^* = \min\{l, r\}$ *and* $\phi = \min\{m^* + l^*, 2l^*\}$.

Proof. As in the proof of Theorem 3.1, we conclude that $\widetilde{x}_n$ exists uniquely from assumption (a), and hence it is meaningful to give assumptions (b) and (c) for $\widetilde{x}_n$. Moreover, it is clear that $\widetilde{x}'_n$ exists and is unique.

Now we have

$$\|x_0 - \widetilde{x}'_n\|_{\infty,\mathcal{D}} \leq \|x_0 - x'_n\|_{\infty,\mathcal{D}} + \|x'_n - \widetilde{x}'_n\|_{\infty,\mathcal{D}},$$

and Theorem 2.2 (ii) shows that $\|x_0 - x'_n\|_{\infty,\mathcal{D}} \leq ch^\sigma$. Hence it suffices to obtain the bound of $\|x'_n - \widetilde{x}'_n\|_{\infty,\mathcal{D}}$. From (1.3) and (1.5), we have

$$\|x'_n - \widetilde{x}'_n\|_{\infty,\mathcal{D}} \leq \|K\widetilde{x}_n - \widetilde{K}\widetilde{x}_n\|_{\infty,\mathcal{D}} + \|Kx_n - K\widetilde{x}_n\|_{\infty,\mathcal{D}}. \tag{3.6}$$

The bound of the first term on the right–hand side of (3.6) follows from assumption (b).

To bound the second term, we rewrite it as follows:

$$Kx_n - K\widetilde{x}_n = K'(x_0)(x_n - \widetilde{x}_n) + s_n + \widetilde{s}_n, \tag{3.7}$$

where $s_n = o(\|x_n - x_0\|)$ and $\widetilde{s}_n = o(\|\widetilde{x}_n - x_0\|)$. We obtain from (3.5) the fact that for any $t \in \bar{\mathcal{D}}$,

$$\begin{aligned} & K'(x_0)(x_n - \widetilde{x}_n)(t) \\ &= \langle l_t, q_n\mathbf{a}_n - q_n\widetilde{\mathbf{a}}_n\rangle \\ &= \langle (I - L^*)g_t, q_n\mathbf{a}_n - q_n\widetilde{\mathbf{a}}_n\rangle \\ &= \langle P_n g_t, (I - L)(q_n\mathbf{a}_n - q_n\widetilde{\mathbf{a}}_n)\rangle \\ &\qquad + \langle (I - P_n)g_t, (I - L)(q_n\mathbf{a}_n - q_n\widetilde{\mathbf{a}}_n)\rangle. \end{aligned} \tag{3.8}$$

From (3.1), (3.2) and some algebraic manipulation, we obtain

$$\begin{aligned} \langle P_n g_t, (I - L)(q_n\mathbf{a}_n - q_n\widetilde{\mathbf{a}}_n)\rangle &= \langle P_n g_t, q_n(r_n f - f_n)\rangle \\ &+ \langle P_n g_t, q_n\{r_n K(q_n\widetilde{\mathbf{a}}_n) - r_n\widetilde{K}(q_n\widetilde{\mathbf{a}}_n)\}\rangle + t_n + \widetilde{t_n}, \end{aligned} \tag{3.9}$$

where $t_n = o(\|x_n - x_0\|)$ and $\widetilde{t_n} = o(\|\widetilde{x}_n - x_0\|)$. Now Lemma 2.4(ii) gives $q_n r_n = P_n$ and hence we have

$$\begin{aligned} \langle P_n g_t, q_n(r_n f - f_n)\rangle &= \langle q_n r_n g_t, q_n(r_n f - f_n)\rangle \\ &= (r_n g_t)^T G_n(r_n f - f_n) \\ &= (r_n g_t)^T(\mathbf{w}_n - \widetilde{\mathbf{w}}_n) \\ &= E_1(P_n g_t, f). \end{aligned}$$

Similarly, we obtain

$$\langle P_n g_t, q_n\{r_n K(q_n\widetilde{\mathbf{a}}_n) - r_n\widetilde{K}(q_n\widetilde{\mathbf{a}}_n)\}\rangle = E_2(P_n g_t, K\widetilde{x}_n).$$

Hence it follows from (3.8) and (3.9) that

$$\begin{aligned} \|Kx_n - K\widetilde{x}_n\|_{\infty,\mathcal{D}} \le \sup_t \{ &|E_1(P_n g_t, f)| + |E_2(P_n g_t, K\widetilde{x}_n)| \\ &+ |\langle g_t - P_n g_t, (I - L)(q_n\mathbf{a}_n - q_n\widetilde{\mathbf{a}}_n)\rangle| \} \\ &+ c_1\|x_n - x_0\|^2_{\infty,\mathcal{D}} + c_2\|\widetilde{x}_n - x_0\|^2_{\infty,\mathcal{D}}. \end{aligned} \tag{3.10}$$

We obtain the bounds of the first two terms in supremum in (3.10) from assumptions (d) and (e). We note that $(I - P_nL)(x_n - \tilde{x}_n) \in \mathcal{U}_n$ for the bound of the third term. Then from (2.1) we have $\langle g_t - P_n g_t, (I - P_nL)(x_n - \tilde{x}_n)\rangle = 0$. Hence we obtain

$$\begin{aligned}\sup_t |\langle g_t - P_n g_t, (I - L)(q_n \mathbf{a}_n - q_n \tilde{\mathbf{a}}_n)\rangle| \\ = \sup_t |\langle g_t - P_n g_t, (P_nL - L)(x_n - \tilde{x}_n)\rangle| \\ \le \sup_t \|g_t - P_n g_t\|_{1,\mathcal{D}} \|P_nL - L\|_{L_\infty} \|x_n - \tilde{x}_n\|_{\infty,\mathcal{D}}.\end{aligned}$$

Since $g_t \in W_1^m(\mathcal{D})$, $\sup_t \|g_t - P_n g_t\|_{1,\mathcal{D}} \le ch^{m^*}$ by the similar proof as that of (2.6). Moreover, we know that L is a bounded operator from $L_\infty(\mathcal{D})$ to $W_\infty^z(\mathcal{D})$ under the assumption on l ([10]), and hence it follows that $\|P_nL - L\|_{L_\infty} \le ch^z$. From assumption (c), we have

$$\sup_t |\langle g_t - P_n g_t, (I - L)(q_n \mathbf{a}_n - q_n \tilde{\mathbf{a}}_n)\rangle| \le ch^{m^*} \cdot c_1 h^z \cdot c_2 h^\zeta \le ch^\sigma.$$

The last two terms of (3.10) are bounded by $2l^*$ and 2ζ, and thus by σ. This completes the proof.

4. Numerical Examples

One can usually determine the orders of convergence of $\tilde{x}_n$ and $\tilde{x}'_n$ when the basic functions and the quadrature technique(s) have been chosen to solve a particular integral equation.

To illustrate our results, we consider the situation in one dimension. Thus we employ the integral equation

$$(4.1) \qquad x(t) = f(t) + \int_a^b k(t, s, x(s))ds \quad \text{for } t \in \bar{\mathcal{D}} = [a, b],$$

where $-\infty < a < b < \infty$. We assume that $f \in W_\infty^{2r}(\mathcal{D})$ and $k \in W_\infty^{2r}(\mathcal{D} \times \mathcal{D} \times R)$, and hence $x \in W_\infty^{2r}(\mathcal{D})$. Since f and k are smooth, we use interpolatory quadrature rules to evaluate the required integrals.

Let $X = \{x_0, \ldots, x_N\}$, where $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$, be a mesh on $[a, b]$ with $h = \max_{1 \le i \le N}(x_i - x_{i-1})$. We assume that the mesh is quasi–uniform, that is, there exists a constant c such that $h/\min_{1\le i\le N}(x_i - x_{i-1}) \le c$. Then for $n = (N-1)(r-\nu)+r$ and $0 \le \nu < r$, we take $\mathcal{U}_n$ to be $S_r^\nu(X)$, the space of piecewise polynomial of degree $\le r-1$ on each (x_{i-1}, x_i) $(1 \le i \le N)$ and has $\nu - 1$ continuous derivatives on (a,b). We note that $n \le ch^{-1}$; the approximation property (2.2) holds for $S_r^\nu(X)$ ([6]); and P_n is a uniformly bounded operator on $L_p(a,b)$ $(1 \le p \le \infty)$ under the quasi–uniformity condition on the mesh ([5]). Moreover, the inverse Gram matrix G_n^{-1} satisfies $\|G_n^{-1}\|_\infty \le ch^{-1}$ ([11]), and hence the value of γ in (2.4) is 1.

Specially, we consider the case when $r = 2$ and $\nu = 1$, and then we take $\{u_i\}_{i=1}^n$ to be the piecewise linear functions. We know that the piecewise linear functions satisfy

$$\|\sum_{i=1}^n |u_i|\|_{\infty,\mathcal{D}} = 1$$

and have derivatives satisfying

$$\|D^j \dot{u}_i\|_{\infty,\mathcal{D}} \le ch^{-j} \qquad (1 \le i \le n,\ j = 0,1). \tag{4.2}$$

Suppose that we approximate the integrals $\langle u_i, K'(x_n)u_j\rangle$, $\langle u_i, f\rangle$ and Kx_n for $1 \le i,j \le n$, by using the $q(\ge 2)$ nodes Gaussian quadrature rule which exact for the polynomials of degree $\le 2q-1$. We employ the product form of the one–dimensional rule for the double integrals $\langle u_i,\ K'(x_n)u_j\rangle$ with $1 \le i,j \le n$. Then we use the Broyden's method ([8]) to solve the nonlinear system appeared in equation (1.4).

Now we examine the orders of convergence of the discrete Galerkin methods for the following equation

$$x(t) = f(t) + \int_0^1 \frac{1}{t+s+x(s)}ds \quad \text{for } 0 \le t \le 1. \tag{4.3}$$

Here f is so chosen that for $t \in [0,1]$, $x_0(t) = e^t$ is the solution of the equation (4.3). In this case, the constants l and m can be chosen as large as desired and $x_0 \in C^{2r}[0,1]$. We have from Theorem 2.2 that $\|x_0 - x_n\|_{\infty,\mathcal{D}} = O(h^2)$ and $\|x_0 - x'_n\|_{\infty,\mathcal{D}} = O(h^4)$. It can be shown that we can take $\mu = \frac{3}{2} > \lambda = 1$ and $\delta = \xi = 2$ and hence we obtain $\|x_0 - \widetilde{x}_n\|_{\infty,\mathcal{D}} = O(h^2)$ in Theorem 3.1 with (4.2). From Theorem 3.2 we have $\eta = \tau = \rho = z + \zeta = 4$, so that $\|x_0 - \widetilde{x}'_n\|_{\infty,\mathcal{D}} = O(h^4)$. Atkinson and Potra [4] obtained the same orders in a different way for the integral equation (4.3) when $x_0 \in C^{2r}[0,1]$. The numerical results are given in Table 1 and Table 2 for $q = 2$ and $q = 4$, respectively.

If we choose f so that $x_0(t) = \exp(|t - \frac{1}{2}|)$ is the exact solution of (4.3), then x_0 and f are not continuously differentiable any more. But $x_0 \in W^{2r}_{\infty}(\mathcal{D})$, and hence we can apply Theorem 3.1 and Theorem 3.2 to (4.3) with l and m sufficiently large enough. The numerical results are given in Table 3 and Table 4 for $q = 2$ and $q = 4$, respectively. Hence we obtain the same orders of convergence of two methods that we have in above case.

N	n	$\|x_0 - \widetilde{x}_n\|_{\infty,\mathcal{D}}$	ratio	$\|x_0 - \widetilde{x}'_n\|_{\infty,\mathcal{D}}$	ratio
2	3	5.100E-2		3.615E-4	
4	5	1.344E-2	3.79	2.684E-5	13.47
8	9	3.452E-3	3.89	1.783E-6	15.05
16	17	8.740E-4	3.95	1.133E-7	15.74
32	33	2.198E-4	3.98	7.151E-9	15.84

Table 1. $x_0(t) = e^t$, q=2;

N	n	$\|x_0 - \widetilde{x}_n\|_{\infty,\mathcal{D}}$	ratio	$\|x_0 - \widetilde{x}'_n\|_{\infty,\mathcal{D}}$	ratio
2	3	5.003E-2		2.975E-4	
4	5	1.331E-2	3.76	2.231E-5	13.34
8	9	3.435E-3	3.87	1.474E-6	15.14
16	17	8.719E-4	3.94	9.375E-8	15.72
32	33	2.196E-4	3.97	5.858E-9	16.00

Table 2. $x_0(t) = e^t$, q=4;

N	n	$\|x_0 - \widetilde{x}_n\|_{\infty,\mathcal{D}}$	ratio	$\|x_0 - \widetilde{x}'_n\|_{\infty,\mathcal{D}}$	ratio
2	3	2.923E-2		7.088E-5	
4	5	8.191E-3	3.57	5.006E-6	14.16
8	9	2.094E-3	3.91	3.165E-7	15.82
16	17	5.301E-4	3.95	1.982E-8	15.97
32	33	1.334E-4	3.97	1.244E-9	15.93

Table 3. $x_0(t) = \exp(|t - \frac{1}{2}|)$, q=2;

N	n	$\|x_0 - \tilde{x}_n\|_{\infty,\mathcal{D}}$	ratio	$\|x_0 - \tilde{x}'_n\|_{\infty,\mathcal{D}}$	ratio
2	3	2.825E-2		7.509E-5	
4	5	8.114E-3	3.48	4.629E-6	16.22
8	9	2.084E-3	3.89	3.018E-7	15.34
16	17	5.288E-4	3.94	1.908E-8	15.82

Table 4. $x_0(t) = \exp(|t - \frac{1}{2}|)$, q=4;

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Department of Mathematics
Yonsei University
Seoul 120-749, Korea

Department of Computer Science
Yonsei University
Seoul 120-749, Korea