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ON THE WEAK LAW OF LARGE NUMBERS FOR ARRAYS OF PAIRWISE INDEPENDENT RANDOM VARIABLES

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Recently Hong and Oh [5] provided a fairly general weak law for arrays in the following form: Let $\{(X_{ni}, 1 \leq i \leq k_n), n \geq 1\}, k_n \to \infty$ as $n \to \infty$, be an array of random variables on $(\Omega, \mathcal{F}, P)$ and set $\mathcal{F}_{nj} = \sigma\{X_{ni}, 1 \leq i \leq j\}, 1 \leq j \leq k_n, n \geq 1$, and $\mathcal{F}_{n_0} = \{\phi, \Omega\}, n \geq 1$. Suppose that $\frac{1}{k_n} \sum_{i=1}^{k_n} aP\{|X_{ni}|^p > a\} \to 0$ as $a \to \infty$ uniformly in n for some $0 < p < 2$. Then $S_n/k_n^{1/p} \to 0$ in probability as $n \to \infty$ where $S_n = \sum_{i=1}^{k_n}(X_{ni} - E(X_{ni}I(|X_{ni}|^p \leq k_n)|\mathcal{F}_{n,i-1}))$.

In this note, we will prove the following result under the same domination condition of Hong and Oh [5].

THEOREM. *Let $\{(X_{ni}, 1 \leq i \leq k_n), n \geq 1\}, k_n \to \infty$ as $n \to \infty$, be an array of pairwise independent random variables and set $S_n = \sum_{i=1}^{k_n} X_{ni}, n \geq 1$. Suppose that for some $0 < p < 2$*

$$(1) \qquad \frac{1}{k_n} \sum_{i=1}^{k_n} aP\{|X_{ni}|^p > a\} \to 0 \quad \text{as} \quad a \to \infty \quad \text{uniformly in } n.$$

Then $(S_n - a_n)/k_n^{1/p} \to 0$ in probability as $n \to \infty$, where $a_n = \sum_{i=1}^{k_n} E(X_{ni}I(|X_{ni}|^p \leq k_n))$, $n \geq 1$.

REMARK. We have different centering from that in Hong and Oh [5] from which we cannot have this result directly.

Proof of Theorem. The proof follows closely from that of Hong and Oh [5]. Namely, set for $1 \leq i \leq k_n, n \geq 1, X'_{ni} = X_{ni}I\{|X_{ni}|^p \leq k_n\}$ and $S'_n = \sum_{i=1}^{k_n} X'_{ni}$. Then, for each $n \geq 2$, $P\{|S_n/k_n^{1/p} - S'_n/k_n^{1/p}| > \varepsilon\}$

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$\leq P\{S_n \neq S'_n\} = P\{\cup_{i=1}^{k_n}\{X_{ni} \neq X'_{ni}\}\} \leq \sum_{i=1}^{k_n} P\{|X_{ni}|^p > k_n\} = \frac{1}{k_n}\sum_{i=1}^{k_n} k_n P\{|X_{ni}|^p > k_n\}$, so that (1) entails $S_n/k_n^{1/p} - S'_n/k_n^{1/p} \to 0$ in probability. Thus to prove the theorem it suffices to verify that

$$\text{(2)} \qquad \frac{S'_n - a_n}{k_n^{1/p}} \longrightarrow 0 \quad \text{in probability.}$$

Since $X'_{ni} - EX'_{ni}$, $1 \leq i \leq k_n$, are pairwise independent and $E(X'_{ni} - E(X'_{ni}))^2 \leq E(X'_{ni})^2$, we have

$$\begin{aligned}
& E(S'_n - \sum_{i=1}^{k_n} E(X'_{ni}))^2 \leq \sum_{i=1}^{k_n} E(X'_{ni})^2 \\
& = \sum_{i=1}^{k_n}\sum_{j=1}^{k_n}\int_{\{j-1<|X_{ni}|^p\leq j\}} X_{ni}^2 dP \\
& \leq \sum_{i=1}^{k_n}\sum_{j=1}^{k_n} j^{2/p}(P\{|X_{ni}|^p > j-1\} - P\{|X_{ni}|^p > j\}) \\
& = \sum_{i=1}^{k_n}[P\{|X_{ni}|^p > 0\} - k_n{}^{2/p}P\{|X_{ni}|^p > k_n\} \\
& \qquad + \sum_{j=1}^{k_n-1}((j+1)^{2/p} - j^{2/p})P\{|X_{ni}|^p > j\}] \\
& \leq k_n + \sum_{j=1}^{k_n}((j+1)^{2/p} - j^{2/p})\sum_{i=1}^{k_n} P\{|X_{ni}|^p > j\} \\
& \leq k_n(1 + c\sum_{j=1}^{k_n}((j+1)^{2/p-1} - j^{2/p-1})k_n^{-1}\sum_{i=1}^{k_n} jP\{|X_{ni}|^p > j\}) \\
& \leq k_n(1 + c\sum_{j=1}^{k_n}((j+1)^{2/p-1} - j^{2/p-1})\sup_n\{k_n^{-1}\sum_{i=1}^{k_n} jP\{|X_{ni}|^p > j\}\}),
\end{aligned}$$

where c is an unimportant positive constant and the second equality comes from Lemma 5.1.1(4) of Chow and Teicher [2]. By the hypothesis (1), $\sup_n\{k_n^{-1}\sum_{i=1}^{k_n} jP\{|X_{ni}|^p > j\}\}$ goes to zero as $j \to \infty$ and

$\sum_{j=1}^{k_n}((j+1)^{\frac{2}{p}-1} - j^{\frac{2}{p}-1}) = (k_n+1)^{\frac{2}{p}-1} - 1$. Thus, by Toeplitz lemma [1],

$$E(S_n' - \sum_{i=1}^{n} E(X_{ni}'))^2 = o(k_n^{2/p}),$$

which implies (2) and hence completes the proof.

COROLLARY ([3, Ex.5.2.12]). *Let $0 < p < 2$ and suppose that $\{X_n\}$ are pairwise independent, identically distributed random variables obeying $nP\{|X_1|^p > n\} = o(1)$. Then*

$$\frac{S_n - nEX_1I(|X_1|^p \le n)}{n^{1/p}} \to 0 \quad \textit{in probability.}$$

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