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EQUIVARIANT VECTOR BUNDLES OVER S^1

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§1. Introduction

Let G be a compact Lie group and let S^1 denote the unit circle in $\mathbb{R}^2$ with the standard metric. Since every smooth compact Lie group action on S^1 is smoothly equivalent to a linear action (cf. [3] TH 2.0), we may think of S^1 with a smooth G-action as $S(V)$ the unit circle of a real 2-dimensional orthogonal G-module V. In this paper we consider smooth G-vector bundles over $S(V)$. We say that a smooth G-vector bundle over $S(V)$ is trivial if it is isomorphic to a product bundle $S(V) \times F \to S(V)$ for some real G-module F. Since the trivial bundles are well understood, we will consider a non-trivial smooth G-vector bundle over $S(V)$ in the following. In this case it turns out there is a real 2-dimensional orthogonal G-module U such that $S(U)/\mathbb{Z}_2 \cong S(V)$ where $\mathbb{Z}_2 = \{\pm 1\}$ and it acts on $S(U)$ as scalar multiplication (see §2). In [2], we proved

PROPOSITION. *A non-trivial smooth G-line bundle over $S(V)$ is isomorphic to $S(U) \times_{\mathbb{Z}_2} \delta \to S(U)/\mathbb{Z}_2 = S(V)$, where δ is a real 1-dimensional $G \times \mathbb{Z}_2$-module and the $\mathbb{Z}_2$-action on it is non-trivial.*

We obtain a similar result for a higher dimensional smooth G-vector bundle over $S(V)$ when the G-action on $S(V)$ is effective, in other words, when V is a faithful representation.

THEOREM A. *A smooth non-trivial G-vector bundle over $S(V)$ is isomorphic to $S(U) \times_{\mathbb{Z}_2} W \to S(U)/\mathbb{Z}_2 = S(V)$ if the G-action on $S(V)$ is effective, where W is the direct sum of real 1-dimensional $G \times \mathbb{Z}_2$-modules and the $\mathbb{Z}_2$-action on W is non-trivial.*

Theorem A follows immediately from the proposition above and the following theorem.

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THEOREM B. *A smooth G-vector bundle over $S(V)$ is isomorphic to the Whitney sum of smooth G-line bundles if the G-action on $S(V)$ is effective.*

Theorem B is not true if we drop the effectiveness assumption. For instance, take a compact Lie group G with an irreducible G-module W of dimension greater than 1. Then the product G-vector bundle $S^1 \times W \to S^1$, where the G-action on S^1 is trivial, does not have a proper smooth G-subbundle, in particular it is not isomorphic to the Whitney sum of smooth G-line bundles.

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§2. Equivariant line bundles over S^1

In this section we give the proof of the proposition in the introduction for the convenience of the reader.

Let $L \to S(V)$ be a non-trivial smooth G-line bundle. Since G is compact, L admits a G invariant fiber metric. We choose one and fix it.

Suppose that the bundle is trivial when we forget the action. Then we can identify L with $S^1 \times \mathbb{R}$ and may assume that the G-action on $S^1 \times \mathbb{R}$ preserves the standard fiber metric on $S^1 \times \mathbb{R}$. Hence one can express the action of $g \in G$ on $S^1 \times \mathbb{R}$ as

$$(x, v) \to (\rho(g)(x), \varphi_g(x)v) \qquad \text{for } (x, v) \in S^1 \times \mathbb{R}$$

where $\rho : G \to O(2)$ is the homomorphism associated with V and $\varphi_g(x)$ is a scalar. Since the action of g preserves the standard metric on $S^1 \times \mathbb{R}$, $\varphi_g(x)$ must be ± 1. The map $\varphi_g : S^1 \to \{\pm 1\} = \mathbb{Z}_2$ is continuous and S^1 is connected, so $\varphi_g(x)$ is independent of $x \in S^1$. Hence we have a homomorphism $\varphi : G \to \mathbb{Z}_2$ given by $g \to \varphi_g$. This shows that the G-line bundle $L \to S(V)$ is trivial.

Thus we may assume that our G-line bundle $L \to S(V)$ remains non-trivial after we forget the action. Then the total space of its sphere bundle, denoted L', is diffeomorphic to S^1 and the projection $\pi : L' \to S(V)$ is an equivariant double covering map. Since any smooth G-action on S^1 is smoothly equivalent to a linear action as mentioned in the introduction, we may assume $L' = S(U)$ for some real 2-dimensional orthogonal G-module U. The G-line bundle induced by π from $L \to S(V)$ is trivial when we forget the action, so the above argument shows that the induced bundle is isomorphic to a product bundle $S(U) \times \delta \to S(U)$ for some real 1-dimensional G-module δ. Therefore our G-line bundle $L \to S(V)$ is isomorphic to $S(U) \times_{\mathbb{Z}_2} \delta \to S(U)$ where $\mathbb{Z}_2$ acts on δ as scalar multiplication. This proves the proposition.

§3. Proof of Theorem B

Since the G-action on the base space $S(V)$ is assumed to be effective, we may think of G as a subgroup of $O(2)$. Consider a normal subgroup $N = G \cap SO(2)$ of G, which is $SO(2)$ or a finite cyclic group. Since the restricted N-action on $S(V)$ is free, taking quotient by the N-action gives a bijective correspondence between G-vector bundles over $S(V)$ and G/N-vector bundles over $S(V)/N$ (cf. [1] TH 1.6.1). Its inverse is given by taking pullback bundles via the quotient map $S(V) \to S(V)/N$. Therefore it suffices to prove that

(*) *any smooth G/N-vector bundle over $S(V)/N$ is isomorphic to the Whitney sum of smooth G/N-line bundles.*

If $N = SO(2)$ (i.e., $G = O(2)$ or $SO(2)$), then $S(V)/N$ is a point, so (*) is obvious in this case. Therefore we may assume that N is finite cyclic in the following. Then $S(V)/N$ is again a circle.

Case 1. Suppose G is a subgroup of $SO(2)$. Then $G = N$, in other words, G/N is trivial. Since any (non-equivariant) smooth vector bundle over S^1 is isomorphic to the Whitney sum of smooth line bundles as is well known, (*) follows.

Case 2. Suppose G is not a subgroup of $SO(2)$. Then G is a dihedral group, $G/N \cong \mathbb{Z}_2$ and the induced G/N-action on $S(V)/N$ is reflection; so (*) reduces to the following lemma.

LEMMA. *Suppose $\mathbb{Z}_2$ acts on S^1 by reflection. Then any smooth $\mathbb{Z}_2$-vector bundle E over S^1 is isomorphic to the Whitney sum of smooth $\mathbb{Z}_2$-line bundles.*

Proof. Let $\{z_0, z_1\}$ be the fixed set of the $\mathbb{Z}_2$-action on S^1. Choose an eigenvector v_i of E at z_i and connect v_0 and v_1 by a smooth path to get a non-zero cross section of E on the upper half circle. Extend it to a cross section of E on S^1 by using the $\mathbb{Z}_2$-action. The resulting cross section on S^1 may not be continuous, but the lines generated by it form a smooth $\mathbb{Z}_2$-line subbundle L of E. So we can decompose $E \cong E' \oplus L$ where E' is a smooth $\mathbb{Z}_2$-vector bundle. Apply the same argument to E' and so on. Then the lemma follows.

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