A NOTE ON THE RANK OF THE FLOW

JOSEPH AUSLANDER AND YOUNG-KEY KIM

1. Preliminaries

In this paper a *flow* will be a pair (X, T) where X is a compact Hausdorff space, and T is a homeomorphism of X onto X. We will usually but not always assume that X is a metric space. We sometimes suppress the homeomorphism T notationally, and just denote a flow by X. General references for the preliminary dynamical notations discussed in this section are [5] and [3]. (The latter should be used with care, since transformations are written on the right there.)

The *trivial* flow is the identity homeomorphism on the one-point space; in this case we write $X = 1$.

If $x \in X$, the *orbit* of x is the set $O(x) = \{T^n(x) \mid n \in Z\}$. A subset Λ of X is a *minimal* set if Λ is nonempty, closed, T-invariant $(T(\Lambda) \subset \Lambda)$, and contains no proper subsets with these properties. It is easy to see that a nonempty subset Λ of X is minimal if and only if it is the orbit closure of each of its points: $\Lambda = \overline{O(x)}$, for all $x \in \Lambda$. If $\overline{O(x)}$ is a minimal set, x is said to be an *almost periodic point* . If (X, T) is itself minimal, (so $X = \overline{O(x)}$ for every $x \in X$), then we say that (X, T) is a *minimal flow*.

If (X, T) and (Y, S) are flows, a *homeomorphism* from X to Y is a continuous equivariant map π, that is, $\pi(T(x)) = S(\pi(x))$, for all $x \in X$. If (Y, S) is minimal, π is necessarily onto.

A flow (X, T) is *equicontinuous* if the powers of $T\{T^n | n \in Z\}$ form an equicontinuous family of maps. Every flow (X, T) has a unique maximal equicontinuous factor - that is, there is an equicontinuous flow (Y, S) which is a factor of (X, T) such that every equicontinuous factor of (X, T) is a factor of (Y, S) [4].

If (X, T) is a flow, the points x and y are *proximal* if there is a sequence $\{n_i\}$ such that $\lim T^{n_i}(x) = \lim T^{n_i}(y)$. (Here and elsewhere

in the paper, when we use sequences in definitions and arguments we are assuming the phase space is metric. In case the space is not metric "sequences" should be replaced by "nets"). Equivalently, x and y are proximal if the diagonal $\Delta_X \subset \overline{O(x,y)}$. We write $P(X,T)$ or just P for the proximal relation: $P = \{(x,y)|x$ and y are proximal$\}$. It is clear that P is an invariant subset of the flow $(X \times X, T \times T)$ and that $P(X,T) = P(X,T^n)$ for $n \neq 0$.

It is also useful to relative the proximal relation. A homeomorphism $\pi : X \to Y$ is said to be proximal if, whenever $\pi(x) = \pi(x')$, then $(x,x') \in P$. In this case, we say that X is a proximal extension of Y.

2. Rank of the flow

Given a flow (X,T) and an index set Γ, we consider the flow (X^Γ, T_Γ) on the product space X^Γ in which T acts on each cardinality and $T_\Gamma : X^\Gamma \to X^\Gamma$ is defined by $[T_\Gamma(z)]_\delta = T(z_\delta)$. We will denoted by $z_\Gamma = \{z_\delta \mid \delta \in \Gamma\}$. It is easy to prove the following remark.

REMARK 2.1. Let Γ and Γ' be any two index sets and let $z \in X^\Gamma$ and $w \in X^{\Gamma'}$ be such that $z_\Gamma = w_{\Gamma'}$. Then z is an almost periodic point of (X^Γ, T_Γ) if and only if w is an almost periodic points of $(X^{\Gamma'}, T_{\Gamma'})$.

DEFINITION 2.2. A subset Λ of X is said to be an *almost periodic set* if there exists an index set Γ and an almost periodic point z of (X^Γ, T_Γ) such that $z_\Gamma = \Lambda$.

DEFINITION 2.3. A subset Λ of X is an *almost periodic set mod T* if it is an almost periodic set and distinct points of Λ lie on distinct orbits that is $x, y \in \Lambda$ and $x = T^j(y)$ implies $j = 0$. Clearly maximal almost periodic sets mod T exist.

REMARK 2.4. Let $z \in X^\Gamma$ and let $w \in \overline{O(z)}$. If $z_\alpha = T^j(z_\delta)$ for some $\alpha, \delta \in \Gamma$, then $w_\alpha = T^j(w_\delta)$.

LEMMA 2.5 ([1]). *All maximal almost periodic sets mod T of the flow (X,T) have the same cardinality.*

By the Lemma 2.5, we can define the *almost periodic rank* (or just rank) of a flow to be the cardinality of a maximal almost periodic set mod T.

In a similar manner we can define the *almost periodic rank of a homomorphism*. Let (X, T) and (Y, S) be flows with (Y, S) minimal and $|Y|$ infinite and let $\pi : X \to Y$ be a homomorphism. The rank of π is defined to be the cardinality of a maximal almost periodic subset of a fiber (that is, rank $\pi = |\Lambda|$, where Λ is an almost periodic subset of $\pi^{-1}(y)$ for $y \in Y$, and Λ is maximal with respect to this property. Since Y is infinite, any such almost periodic set is necessarily almost periodic mod T).

The following theorem certify that the rank of π is well defined.

THEOREM 2.6. *All maximal almost periodic subsets of the fibers of π have the same cardinality.*

Proof. Let Λ and Λ' be maximal almost periodic sets of the fibers $\pi^{-1}(y)$ and $\pi^{-1}(y')$. If $y = y'$, choose index sets Γ and Γ' and points $z \in X^{\Gamma}$ and $z' \in X^{\Gamma'}$ such that $\Gamma \cap \Gamma' = \phi$, $z_{\Gamma} = \Lambda$ and $z'_{\Gamma'} = \Lambda'$.

Consider $(z, z') \in X^{\Gamma} \times X^{\Gamma'}$ and let $T_{\Gamma} \times T_{\Gamma'}$ act on this space. There exists a $T_{\Gamma} \times T_{\Gamma'}$ minimal set Y in the orbit closure of (z, z') which projects onto $\overline{O(z)}$. Hence there exists a $T_{\Gamma} \times T_{\Gamma'}$ almost periodic point of the form (z, w). Set $\Lambda_0 = w_{\Gamma'}$. Since $w \in \overline{O(z')}, \Lambda_0$ is a maximal almost periodic set mod T and $|\Lambda'| = |\Lambda_0|$ and Λ_0 is a maximal almost periodic set, $|\Lambda'| = |\Lambda_0|$ and $\Lambda_0 \subset \pi^{-1}(y)$. Because $\Lambda \cap \Lambda_0$ is an almost periodic set, $|\Lambda| = |\Lambda_0|$ by maximality. If $y \neq y'$, choose $z'' \in \overline{O(z')}$ such that $\Lambda'' = z''_{\Gamma'} \subset \pi^{-1}(y)$ and Λ'' is a maximal almost periodic subset of $\pi^{-1}(y)$. Then $|\Lambda'| = |\Lambda''|$ and by the above $|\Lambda''| = |\Lambda|$.

LEMMA 2.7 ([5]). *Let (X, T) and (Y, S) be minimal flows and let $\varphi : X \to Y$ a homomorphism. Then the following are equivalent :*
  (a) *There exist $x, y \in X$, with $x \neq y$, and $\varphi(x) = \varphi(y)$ such that (x, y) is an almost periodic point of $(X \times X, T \times T)$*
  (b) *There exist $x, y \in X$, with $\varphi(x) = \varphi(y)$ such that $(x, y) \notin P(X, T)$.*

We have the following theorem by definition.

THEOREM 2.8. *Let (X, T) and (Y, S) be infinite minimal flows and let $\pi : X \to Y$ be a homomorphism. Then*
  (i) *rank $(X, T) = ($rank $\pi)$rank (Y, S)*
  (ii) *rank $(Y, S) \leq$ rank (X, T)*

We will state main theorem of this paper.

THEOREM 2.9. *Let (X, T) be a totally minimal flow and let (Y, T) be an infinite minimal flow. Let $\varphi : (X, T) \to (Y, T)$ be a homomorphism. If (X, T) is a graphic, then rank $\varphi = 1$.*

Proof. At first, we will show that φ is proximal homomorphism. Suppose ϕ is not proximal. Then there exist $x, x' \in X$ with $x \neq x'$ and $\varphi(x) = \varphi(x')$ such that $(x, x') \notin P(X, T)$. By Lemma 2.7, there exist $x, x' \in X$ with $x \neq x'$ and $\varphi(x) = \varphi(x')$ such that (x, x') is an almost periodic point of $(X \times X, T \times T)$. Since (X, T) is graphic, $x' = T^m(x)$ for some $m \neq 0$. So $\varphi(x') = \varphi(T^m(x)) = \varphi(x)$. Since (X, T) is totally minimal, $X = \overline{\{T^n(x) \mid n \in Z\}}$. Let $z \in X$. Then there exists a net $\{n_i\}$ in Z such that $T^{n_i}(x) \to Z$ and so $\varphi(T^{n_i}(x)) \to \varphi(z)$. Hence any $z \in X$, $\varphi(z) = \varphi(x)$. This implies Y is trivial. This is a contradiction because Y is infinite. Therefore φ is proximal. Since φ is proximal, let $x, x' \in \varphi^{-1}(y)$ with $x \neq x'$, then (x, x') is not an almost periodic point of $(X \times X, T \times T)$ because $\Delta \subset \overline{O(x, x')}$, so rank φ must be 1.

References

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Department of Mathematics
University of Maryland
College Park, MD20742 U.S.A.

Department of Mathematics
Myong Ji University
Yong-In 449-728, Korea