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ON COMPACT GENERIC SUBMANIFOLDS IN A SASAKIAN SPACE FORM

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Introduction

One of typical submanifolds of a Sasakian manifold is the so-called *generic submanifolds* which are defined as follows: Let M be a submanifold of a Sasakian manifold $\tilde{M}$ with almost contact metric structure (ϕ, G, ξ) such that M is tangent to the structure vector ξ. If each normal space is mapped into the tangent space under the action of ϕ, M is called a generic submanifold of $\tilde{M}$ [2], [8].

Many subjects for generic submanifolds of a Sasakian manifold were investigated from various different points of view. In [4], [6], [7] Ki, Kon and Yano studied basic properties of generic submanifolds in a Sasakian manifold. In particular, under the assumption that the shape operator A^* in the direction of the mean curvature vector is commutative with the f-structure f induced on the submanifold, some characterizations and some classifications of generic submanifolds with parallel mean curvature vector in a Sasakian space form were obtained [1], [3].

On the other hand, Ki, Takagi and Takahashi proved the following theorem:

THEOREM A ([5]). *Let M be an n-dimensional compact generic submanifold of a complex projective space P_mC with nonvanishing parallel mean curvature vector H. If $\|H\|^2 < \frac{1}{n}(\sqrt{n}-1)$, then the shape operator A^* in the direction of the mean curvature vector and the f-structure f induced on M commutes each other. Moreover we have $\|\nabla A^*\|^2 = 8(n-m)$.*

The purpose of this paper is to prove Theorem A as Sasakian analogue.

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1. Generic submanifolds of a Sasakian manifold

In this section, basic properties of generic submanifolds of a Sasakian manifold are recalled [3], [8].

Let $\tilde{M}$ be a Sasakian manifold of dimension $2m+1$ with almost contact metric structure (ϕ, G, ξ). Then for any vector fields X and Y on $\tilde{M}$, we have

$$\phi^2 X = -X + \eta(X)\xi, G(\phi X, \phi Y) = G(X, Y) - \eta(X)\eta(Y),$$
$$\eta(\phi X) = 0, \ \phi\xi = 0, \ \eta(\xi) = 1, \ G(X, \xi) = \eta(X).$$

Furthermore we have

$$\tilde{\nabla}_X \xi = \phi X, \ (\tilde{\nabla}_X \phi)Y = -G(X, Y)\xi + \eta(Y)X, \tag{1.1}$$

where $\tilde{\nabla}$ denotes the Riemannian connection of $\tilde{M}$.

Let M be an $(n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and isometrically immersed in $\tilde{M}$ by the immersion $i : M \to \tilde{M}$. Throughout this paper the indices $i, j, k, \ldots$ run from 1 to $n+1$. We represent the immersion i locally by

$$y^A = y^A(x^h), \ (A = 1, \ldots, n+1, \ldots, 2m+1)$$

and put $B_j^A = \partial_j y^A$, $(\partial_j = \frac{\partial}{\partial x^j})$ then $B_j = (B_j^A)$ are $(n+1)$-linearly independent local tangent vector fields of M. Hereafter the indices $u, v, w, x \ldots$ run from $n+2$ to $2m+1$ and the summation convention will be used. The immersion being isometric, the induced Riemannian metric tensor g with components g_{ji} and the metric tensor δ with components δ_{yx} of the normal bundle are respectively obtained:

$$g_{ji} = G(B_j, B_i), \quad \delta_{yx} = G(C_y, C_x).$$

By denoting ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to g and G, the equations of Gauss and Weingarten for the submanifold M are respectively given by

$$\nabla_j B_i = A_{ji}^x C_x, \quad \nabla_j C_x = -A_{jx}^h B_h, \tag{1.2}$$

where A^x_{ji} are components of the second fundamental tensors and the shape operator A^x in the direction of C_x are related by

$$A^x = (A_j^{hx}) = (A_{jiy} g^{ih} \delta^{yx}), \quad g^{ji} = (g_{ji})^{-1}.$$

An $(n+1)$-dimensional submanifold M, tangent to the structure vector ξ, of a Sasakian manifold $\tilde{M}$ is called a *generic submanifold* if

$$\phi N_P(M) \subset T_P(M)$$

at each point $p \in M$, where $T_P(M)$ is the tangent space of M at p and $N_P(M)$ the normal space at p, [3], [7].

In the following, we have only to consider generic submanifolds of a Sasakian manifold. Then the transforms of B_i and C_x by ϕ are respectively represented in each coordinate neighborhood as follows:

$$\phi B_j = f_j^h B - J_j^x C_x, \quad \phi C_x = J_x^h B_h, \tag{1.3}$$

where we have put $f_{ji} = G(\phi B_j, B_i)$, $J_{jx} = -G(\phi B_j, C_x)$, $J_{xj} = G(\phi C_x, B_j)$, $f_j^h = f_{ji} g^{ih}$ and $J_j^x = J_{jy} \delta^{yx}$. From these definitions we verify that $f_{ji} + f_{ij} = 0$ and $J_{jx} = J_{xj}$.

Since the submanifold M is tangent to the structure vector ξ, we can put

$$\xi = v^h B_h, \tag{1.4}$$

where $v_i = G(C_x, \xi), v^h$ being the associated vector with respect to v_h.

By the properties of the Sasakian structure tensor (ϕ, G, ξ), it is, using (1.3) and (1.4), seen that

$$f_j^t f_t^h = -\delta_j^h + v_j v^h + J_j^x J_x^h, \quad J_y^t J_t^x = \delta_y^x, \tag{1.5}$$

$$f_t^h J_x^t = 0, \quad f_{jt} v^t = 0, \quad J_t^x v^t = 0, \tag{1.6}$$

$$v_t v^t = 1. \tag{1.7}$$

Because of (1.5) and (1.6), it is clear that $f^3 + f = 0$. This shows that f is an f-structure induced on M.

Differentiating (1.3) and (1.4) covariantly along M and taking account of (1.1) and (1.2), we find [6]

$$\nabla_k f_j^h = -g_{kj}v^h + \delta_k^h v_j + A_{kj}^x J_x^h - A_k^{hx} J_{jx}, \tag{1.8}$$

$$\nabla_k J_{jx} = A_{krx} f_j^r, \quad \nabla_j v_i = f_{ji}, \tag{1.9}$$

$$A_{jrx} J^{ry} = A_j^{ry} J_{rx}, \tag{1.10}$$

$$A_{jrx} v^r = -J_{jx}. \tag{1.11}$$

In what follows we suppose that the ambient Sasakian manifold $\tilde{M}$ is of constant ϕ-holomorphic sectional curvature c and of real dimension $2m+1$, which is called a *Sasakian space form*, and is denoted by $\tilde{M}^{2m+1}(c)$. Then the curvature tensor $\tilde{R}$ of $\tilde{M}^{2m+1}(c)$ is given by

$$\begin{aligned}\tilde{R}_{DCBA} = &\frac{1}{4}(c+3)(G_{DA}G_{CB} - G_{DB}G_{CA}) \\ &+ \frac{1}{4}(c-1)(\xi_C\xi_A G_{DB} - \xi_C\xi_B G_{DA} + \xi_D\xi_B G_{CA} - \xi_D\xi_A G_{CB} \\ &+ \phi_{DA}\phi_{CB} - \phi_{DB}\phi_{CA} - 2\phi_{DC}\phi_{BA}).\end{aligned}$$

Thus, we see, using (1.3) and (1.4), that equations of the Gauss, Codazzi and Ricci for M are respectively obtained:

$$\begin{aligned}R_{kjih} = &\frac{1}{4}(c+3)(g_{kh}g_{ji} - g_{jh}g_{ki}) + A_{kh}^x A_{jix} - A_{jh}^x A_{kix} \\ &+ \frac{1}{4}(c-1)(v_k v_i g_{kh} - v_j v_i g_{kh} + v_j v_h g_{ki} - v_k v_h g_{ji} \\ &+ f_{kh}f_{ji} - f_{jh}f_{ki} - 2f_{kj}f_{ih}),\end{aligned} \tag{1.12}$$

$$\nabla_k A_{ji}^x - \nabla_j A_{ki}^x = \frac{1}{4}(c-1)\{J_j^x f_{ki} - J_k^x f_{ki} + 2J_i^x f_{kj}\}, \tag{1.13}$$

$$R_{jixy} = \frac{1}{4}(c-1)(J_{jy}J_{ix} - J_{jx}J_{iy}) + A_{jry}A_{ix}^r - A_{iry}A_{yx}^r, \tag{1.14}$$

where R_{kjih} and R_{jixy} are components of the Riemannian curvature tensor of M and that with respect to the connection induced in the normal bundle of M respectively. We see from (1.12) that the Ricci tensor S with components S_{ji} of M can be expressed as follows:

$$(1.15)\quad S_{ji} = \frac{1}{4}\{(c+3)n + 2(c-1)\}g_{ji} - \frac{1}{4}(c-1)\{(n+2)v_j v_i + 3J_j^z J_{iz}\} + h^x A_{jix} - A_j^{rx} A_{irx}$$

because of (1.5), where $h^x = TrA^x$.

From now on the index $n+2$ will be denoted by the symbol $*$.

Because of (1.8), (1.9) and (1.11), we then have

$$(1.16)\quad \nabla_k \nabla_j J_{i*} = (\nabla_k A_{jr*}) f_i^r + g_{ki} J_{j*} + A_{kj*} v_i + A_{ki}^x A_{jr*} J_x^r - A_{jr*} A_k^{rx} J_{ix}.$$

By using (1.5), (1.6), (1.11) and (1.13), it leads to

$$(1.17)\quad \nabla^i \nabla_j J_{i*} = h^x A_{ir*} J_x^r - A_{ir*} A_s^{rx} J_x^s + \frac{1}{4}(c-1)(n-p)J_{i*} + nJ_{i*},$$

where $p = \operatorname{codim} M$.

2. Compact generic submanifolds

In this section, we consider that a generic submanifold M of a Sasakian space form $\tilde{M}^{2m+1}(c)$. Let H be a mean curvature vector of a generic submanifold M. Namely, it is defined by

$$H = g^{ji} A_{ji}^x C_x/(n+1) = h^x C_x/(n+1),$$

which is independent of the choice of the local field of orthonormal frames $\{C_x\}$.

In what follows we suppose that the mean curvature vector H of M is nonzero and is parallel in the normal bundle. Then we may choose a local field $\{C_x\}$ in such a way that $H = \sigma C_{n+2} = \sigma C_*$, where $\sigma = \|H\|$ is nonzero constant. Because of the choice of the local field, the parallelism of H yields

$$(2.1)\quad \begin{cases} h^x = 0, & x \geq n+3 \\ h^* = (n+1)\|H\|. & \end{cases}$$

The parallelism of the mean curvature vector yields that $R_{jix*} = 0$ because $\|H\| \neq 0$. Thus (1.14) implies

$$A^x_{jr}A^{r*}_i - A^x_{ir}A^{r*}_j = \frac{1}{4}(c-1)(J^x_i J^*_j - J^x_j J^*_i) \tag{2.2}$$

because of (2.1).

Applying (1.20) by J^{j*} and using (2.1), we find

$$\begin{aligned} J^{j*}(\nabla^i\nabla_j J_{i*}) = (n+1)\|H\|A_{ji*}J^j_* J^i_* - (A_{jr*}J^j_*)(A^{rx}_s J^s_x) \\ + \frac{1}{4}(c-1)(n-p) + n, \end{aligned}$$

which together with (1.6) and (2.2) implies that

$$\begin{aligned} J^{j*}(\nabla^i\nabla_j J_{i*}) = (n+1)\|H\|A_{ji*}J^j_* J^i_* - (A_{jrx}J^j_*)(A^{r*}_i J^i_x) \\ + \frac{1}{4}(c-1)(n-1) + n. \end{aligned} \tag{2.3}$$

On the other hand, from (1.19) we can get

$$J^i_*\Delta J_{i*} = -h_{(2)} + 1 + (A_{jrx}J^j_*)(A^{r*}_i J^i_x) \tag{2.4}$$

because of (1.5), (1.11) and (1.13), where $\Delta = g^{ji}\nabla_j\nabla_i$ and $h_{(2)} = A_{ji*}A^{ji*}$.

Now, let us put $U_j = J^{i*}\nabla_j J_{i*} + J^{i*}\nabla_i J_{j*}$. Then the divergence of the vector U is given by

$$\operatorname{div} U = \frac{1}{2}\|\nabla_j J_{i*} + \nabla_i J_{j*}\|^2 + J^{i*}\Delta J_{i*} + J^{j*}\nabla^i\nabla_j J_{i*}.$$

Substituting the first equation of (1.9) and (2.3), (2.4) into this, we obtain

$$\begin{aligned} \operatorname{div} U = \frac{1}{2}\|A_{jr*}f^r_i + A_{ir*}f^r_j\|^2 - h_{(2)} + (n+1)\|H\|A_{ji*}J^j_* J^i_* \\ + \frac{1}{4}(c+3)(n-1) + 2. \end{aligned} \tag{2.5}$$

Making use of the second equation of (1.5), it is seen that

$$\{A_{ji*}J_*^j J_*^i\}^2 \leq h_{(2)}.$$

Hence we get

$$\begin{aligned} & -h_{(2)} + (n+1)\|H\|A_{ji*}J_*^j J_*^i + \frac{1}{4}(c+3)(n-1) + 2 \\ & \geq -h_{(2)} - (n+1)\|H\|\sqrt{h_{(2)}} + \frac{1}{4}(c+3)(n-1) + 2 \\ & \geq \frac{1}{4}(c+3)(n-1) + 2 - (n+1)\|H\|^2(\sqrt{n+1}+1) \end{aligned}$$

because we have in general $(n+1)\|H\|^2 \leq h_{(2)}$.

According to (2.5) and above inequality we have

THEOREM 1. *Let M be an $(n+1)$-dimensional compact generic submanifold of a Sasakian space form $\tilde{M}^{2m+1}(c)$ with nonvanishing parallel mean curvature vector H of M. If*

$$(\sqrt{n+1}+1)(n+1)\|H\|^2 \leq \frac{1}{4}(c+3)(n-1) + 2,$$

then $A^*f = fA^*$.

THEOREM 2. *Let M be an $(n+1)$-dimensional compact generic submanifold of an odd-dimensional unit sphere $S^{2m+1}(1)$ with nonvanishing parallel mean curvature vector H. If $\|H\|^2 \leq \frac{1}{n}(\sqrt{n+1}-1)$, then we have $A^*f = fA^*$ and $\nabla A^* = 0$.*

Proof. By (1.5) and $A^*f = fA^*$, we have

$$(A_{jr*}J_z^r)J_i^z - (A_{ir*}J_z^r)J_j^z + J_{i*}v_j - J_{j*}v_i = 0,$$

which together with (1.5) and (1.6) gives

$$A_{jr*}J_x^r = P_{xz*}J_j^z - \delta_{x*}v_j, \tag{2.6}$$

where we have defined $P_{xz*} = A_{ji*}J_x^j J_z^i$.

Differentiating $A^*f - fA^* = 0$ covariantly and making use of (1.11) and (2.6), we find

$$(\nabla_k A_{jr*})f_i^r + (\nabla_k A_{ir*})f_j^r + g_{kj}J_{i*} + g_{ki}J_{j*}$$
$$= A_k^{rx}(A_{ir*}J_{jx} + A_{jr*}J_{ix}) - P_{xz*}(A_{kj}^x J_j^z + A_{ki}^x J_j^z),$$

or using (1.13) and (2.2) with $c = 1$,

$$(\nabla_k A_{ir*})f_j^r + (g_{ki}\delta_{y*} + P_{xy*}A_{ki}^x - A_i^{r*}A_{kry})J_j^y = 0.$$

Applying this by J_z^j and taking account of (1.5) and (1.6), we obtain

$$A_j^{r*}A_{irz} = \delta_{x*}g_{ji} + P_{yz*}A_{ji}^y. \tag{2.7}$$

The parallelism of the mean curvature vector yields that the restricted Laplacian ΔA_{ji}^* of A^* is given by

$$\Delta A_{ji}^* = S_{jr}A_i^{r*} - R_{kjih}A^{kh*}.$$

Thus, by using (1.12), (1.18) with $c = 1$ and (2.7), it follows that we have $A^{ji*}\Delta A_{ji}^* = 0$. M being compact, it is seen that $\nabla A^* = 0$ since we have in general

$$\frac{1}{2}\Delta h_{(2)} = A^{ji*}\Delta A_{ji*} + \|\nabla A^*\|^2.$$

This complete the proof.

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