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THE STRUCTURE CONFORMAL VECTOR FIELDS ON A SASAKIAN MANIFOLD

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I. Introduction

Let $M(f,\eta,\xi,g)$ be a $(2m+1)$-dimensional Sasakian manifold with soldering form $dp \in \Gamma\mathrm{Hom}(\Lambda^q TM, TM)$ (dp: canonical vector-valued 1-form) where f,η,ξ and g are the (1,1)-tensor field, the structure 1-form, the structure vector field and the metric tensor of M, respectively. Since one may write $\nabla\xi = f dp$, we give the following definition : Any vector field U such that

$$\nabla U = \rho dp + \lambda \nabla\xi\,;\ \rho,\lambda \in c^\infty M, \tag{1.1}$$

is defined as a conformal vector field ((1.1) implies $\mathcal{L}_U g = 2\rho g$)

In III, it is proved that the existence of U on $M(f,\eta,\xi,g)$ is determined by an exterior differential system in involution (in the sense of É. Cartan [3]), and that any M which carries a vector field U, is foliated by autoparallel three-dimensional submanifold of scalar curvature $+1$, tangent to U, fU and ξ. Besides such a Sasakian manifold possesses the remarkable property to be isometric to a unit sphere in a $(2m+2)$-dimensional Euclidean space [6].

Furthermore, any U is an exterior concurrent vector field (see [8]) and of conformal weight $\frac{2m+1}{m}$ [5].

Consider a K-contact manifold $M(f,\eta,\xi,g)$, i.e., a contact metric manifold whose structure vector ξ is a Killing vector field [11].

We give the following definition : Any vector field X such that

$$\mathcal{L}_X\Omega = h\Omega + \gamma\wedge\eta \tag{1.2}$$

where $\Omega = \frac{1}{2}d\eta$, $h \in C^\infty M$, $\gamma \in \Lambda^1 M$, is called an infinitesimal quasi-conformal contact transformation of Ω.

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II. Preliminaries

Let (M, g) be an orientable C^∞-Riemannian manifold and let ∇ be the covariant differential operator defined by the metric tensor g.

Let $\Gamma(TM) : \mathcal{X}M$ be the set of sections of the tangent bundle TM and $\alpha : TM \to T^*M$ be the musical isomorphism [7] defined by g.

If, following [7], we denote by

$$A^q(M, TM) = \Gamma\mathrm{Hom}(\Lambda^q TM, TM)$$

the set of vector-valued q-forms, $q < \dim M$, then

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM)$$

means the exterior covariant derivative operator with respect to ∇. It should be noticed that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike d^2.

If $dp \in A^1(M, TM)$ denotes the soldering form of M, any vector field X such that

$$d^\nabla(\nabla X) = \nabla^2 X = \pi \wedge dp \in A^2(M, TM), \tag{2.1}$$

is defined as exterior concurrent (abbreviation :E.C.)(see [8]).

It has been proved [8] that π is necessarily given by

$$\pi = v\alpha(X); v \neq 0, \tag{2.2}$$

where $v \in C^\infty M$ is the conformal scalar associated with X.

If $\mathcal{R}$ denotes the Ricci tensor of ∇, it follows from (2.1) and (2.2) that

$$\mathcal{R}(X, Z) = -(n-1)\, vg(X, Z) \Rightarrow v = -\frac{1}{n-1}\mathrm{Ric}X,$$

where $Z \in \mathcal{X}M$ and $\dim M = n$.

Let $\mathcal{T} \in \mathcal{X}M$ be any conformal vector field on M (or conformal Killing vector field), that is

$$\mathcal{L}_\mathcal{T} g = 2\rho g \Leftrightarrow \langle \nabla_Z \mathcal{T}, Z' \rangle + \langle \nabla_{Z'} \mathcal{T}, Z \rangle = 2\rho \langle Z, Z' \rangle \tag{2.3}$$

where $\rho \in C^\infty M; Z, Z' \in \mathcal{X}M$ and

$$\rho = \frac{\text{div } \mathcal{T}}{n} \tag{2.4}$$

vector field other than zero vector field does not exist.

Any vector field X such that

$$\mathcal{L}_X \alpha(X) = c(\text{div } X)\alpha(X); \qquad c = \text{const} \tag{2.5}$$

is defined as a self conformal vector field [9].

We also recall the following theorem due to M. Obata [6] (see also [2]): In order that a gradient vector field grad ϕ be an infinitesimal concircular transformation on an n-dimensional manifold M, it is necessary and sufficient that

$$\langle \nabla_Z \text{grad } \phi, Z' \rangle = v \langle Z, Z' \rangle, \qquad Z, Z' \in M, \tag{2.6}$$

where v is a non-vanishing scalar. If $v = -c^2\phi$, then M is isometric to a sphere S^n of radius $\frac{1}{c}$ in an $(n+1)$-dimensional Euclidean space.

In general (even if X is not a conformal vector field) $cn (n = \dim M)$ is called the conformal weight of $\alpha(X)$ (cf. [5]).

III. A Structure Conformal Vector Field on a Sasakian Manifold

Let $M(f, \eta, \xi, g)$ be a $(2m+1)$-dimensional contact metric manifold. In such a manifold the structure tensors f, η and ξ satisfy the equations;

$$\begin{aligned} & f\xi = 0, \qquad \eta(\xi) = 1, \\ & f^2 = -I + \eta \otimes \xi, \qquad \eta(Z) = g(\xi, Z), \\ & g(fZ, fZ') = g(Z, Z') - \eta(Z)\eta(Z'), \\ & g(fZ, Z') = \frac{1}{2} d\eta(Z, Z'), \qquad Z, Z' \in \mathcal{X}M (\text{cf. [11]}). \end{aligned} \tag{3.1}$$

The f-Lie derivative is defined by

$$(\nabla f)Z = \nabla fZ - f\nabla Z, \tag{3.2}$$

and it has been shown in [1] that ξ is a Killing vector fields if and only if $\mathcal{L}_\xi f$ vanishes. In this case M is called a K-contact manifold. A K-contact manifold for which one has

$$(3.3) \qquad (\nabla_Z f)Z' = -g(Z,Z')\xi + \eta(Z')Z,$$

is called a Sasakian manifold.

If M is a Sasakian manifold, then ξ is always E.C. and

$$(3.4) \qquad \nabla^2\xi = -\eta \wedge dp \Rightarrow \mathcal{R}(\xi, Z) = 2mg(\xi, Z)$$

(cf.[7]). Moreover, any E.C.vector field X satisfies

$$(3.5) \qquad \nabla^2 X = -\alpha(X) \wedge dp,$$

and the property of the exterior concurrency is invariant under the action of f(i.e., $\nabla^2 fX = -\alpha(fX) \wedge dp$).

In the more general case when M is a K-contact manifold, we introduce the following two definitions

[i] A vector field U on M such that

$$(3.6) \qquad \nabla U = \rho\, dp + \lambda \nabla\xi\,; \qquad \rho, \lambda \in c^\infty M,$$

is defined as a structure conformal vector field. Effectively, since ξ is a Killing vector field, it is easy to see that the equation (3.6) satisfies the conformal equation, that is (see (2.3)):

$$\mathcal{L}_U g = 2\rho g \Leftrightarrow \langle \nabla_Z U, Z'\rangle + \langle \nabla'_Z U, Z\rangle = 2\rho\langle Z, Z'\rangle,$$

where $Z, Z' \in \mathcal{X}M$, and this implies (see (2.4))

$$\operatorname{div} U = (2m+1)\rho$$

[ii] Any vector field X such that

$$(3.7) \qquad \mathcal{L}_X\Omega = f\Omega + \gamma \wedge \eta; \qquad \gamma \in \Lambda^1 M,\ \phi \in c^\infty M,$$

is called an infinitesimal quasi-conformal contact transformation of Ω (abbreviation : i.q.c.c.t)

Denote by $\mu : TM \to T^*M$, $X \to i_X\Omega$ the bundle isomorphism defined by Ω. If u is any 1-form on M such that du is equated by the second member of (3.7), then clearly $\mu^{-1}(u)$ defines an i.q.c.c.t.

From now on we shall be concerned with Sasakian manifold carrying a structure conformal vector field U.

Now let

$$\mathcal{O} = \text{vect.}\{e_i, fe_i = e_i^*, e_0 = \xi \,|\, i = 1, \ldots, m\,;\, i^* = i + m\}$$

be an adapted local field of orthonormal frames on M and let

$$\mathcal{O}^* = \text{covect.}\{w^i, w^{i^*}, w^0 = \eta\}$$

be its associated coframe field.

Then the soldering form dp and É. Cartan's structure equations are :

$$dp = w^A \otimes e_A\,;\, A \in \{i, i^*, 0\}, \nabla e = \theta \otimes e \tag{3.8}$$

In the above equation θ is the local connection form in the bundle $\mathcal{O}(M)$.

Further since M is Sasakian, by (3.1), (3.3) and (3.8), we have

$$\theta^i_j = \theta^{i^*}_{j^*}, \qquad \theta^{i^*}_j = \theta^{j^*}_i \tag{3.9}$$

Now in order to make simplifications, we set

$$\| U \|^2 = 2l,\ \alpha(U) = t,\ \alpha(fU) = s = i_U\Omega \tag{3.10}$$

and notice that one has

$$s = -\langle U, \nabla\xi\rangle$$

Next, with the help of (3.1) and (3.8), we obtain from (3.6) that

$$dl = \rho t - \lambda s, \tag{3.11}$$

$$d\eta(U) = \rho\eta - s \tag{3.12}$$

and

$$dt = 2\lambda\Omega \Rightarrow \lambda = \text{const.} \tag{3.13}$$

By (3.12), one gets at once

$$ds = d\rho \wedge \eta + 2\rho\Omega, \tag{3.14}$$

and by (3.10) the equation (3.14) implies

$$\mathcal{L}_U \Omega = 2\rho\Omega + d\rho \wedge \eta \tag{3.15}$$

On the other hand, taking account of (3.4), one derives from (3.6) by covariant differentiation

$$\nabla^2 U = -(\lambda\eta - d\rho) \wedge p \tag{3.16}$$

The equation (3.16) proves that any structure conformal vector field on a Sasakian manifold is E.C.

Using (3.5), we find

$$t = \alpha(U) = \lambda\eta - d\rho, \tag{3.17}$$

and we notice that the equation (3.17) is consistent with (3.13).

Denote now by $\sum$ the exterior differential system which defines the structure conformal vector field U. Then, by (3.11), (3.12), (3.13), (3.14) and (3.17), we see that the characteristic number of $\sum$ (see [3]) are $r = 5$, $s_o = 3$, $s_1 = 2$. Consequently, following É. Cartan's test [3], we conclude that $\sum$ is in involution and depends on two arbitrary functions of one argument. Further, by (3.3) and (3.6), one derives

$$\nabla f U = (\eta(U) - \lambda)dp + \rho\nabla\xi + d\rho \otimes \xi \tag{3.18}$$

Next, taking account of (3.17) and $\operatorname{div} Z = \operatorname{tr} \nabla Z$, one finds

$$\operatorname{div} f\, U = 2m(\eta(U) - \lambda) \tag{3.19}$$

We will outline the following property connected with this subject. First, by (3.17), the equation (3.14) becomes

$$ds = \eta \wedge t + 2\rho\Omega,$$

and by (3.1), one has

$$i_{fU}\Omega = \alpha(f^2U) = (\frac{2m+1}{m})\eta(U)\eta - t$$

Then, taking account of (3.19), one may write

$$\mathcal{L}_{fU}s = \frac{1}{m}s(\operatorname{div} fU) = \frac{2m+1}{m}\frac{(\operatorname{div} fU)s}{\dim M} \tag{3.20}$$

Hence, by definition (2.5), the equation (3.20) proves the following salient property: The structure conformal vector field U on a $(2m+1)$-dimensional Sasakian manifold M, turns out, under the action of f, to a self-conformal vector field of conformal weight $\frac{2m+1}{m}$

Denote now by $\mathcal{D}_U = \{U, fU, \xi\}$ the $\mathcal{D}$-distribution defined by U, fU and ξ. Then, if $X_U, X'_U \in \mathcal{D}_U$ are any vector fields of $\mathcal{D}_U$, it is easy to see by (3.1), (3.6) and (3.18), that one has $\nabla_{X'_U} X_U \in \mathcal{D}_U$ which expresses the fact that $\mathcal{D}_U$ is an autoparallel foliation (cf. [4]). On the other hand, since ξ, U and fU, ξ and E.C. vector fields, it follows, by linearity that any vector field X_U of $\mathcal{D}_U$ is E.C. As a consequence of this fact and the results of [8], we conclude that the leaf M_U of $\mathcal{D}_U$ is an autoparallel submanifold of scalar curvature $+1$ of the Sasakian manifold M under consideration.

Next, from (3.17) it follows

$$\operatorname{grad}\rho = \lambda\xi - U$$

and taking account of (3.6) one gets at once

$$\nabla \operatorname{grad}\rho = -\rho\, dp \tag{3.21}$$

which shows that grad ρ is a concurrent vector field [10]. From (3.21) one gets instantly

$$\langle \nabla_Z \operatorname{grad}\rho, Z'\rangle = -\rho\langle Z, Z'\rangle,$$

Applying Obata's theorem (see (2.6)), we obtain that the Sasakian manifold under consideration enjoys the remarkable property to be isometric to a unit sphere in a $(2m+2)$-dimensional Euclidean space.

Thus, we proved the following theorem:

THEOREM 3.1. *Any Sasakian manifold $M(f,\eta,\xi,g)$ which carries a structure conformal vector field U is foliated by autoparallel 3-dimensional submanifolds of scalar curvature $+1$ tangent to U, fU and ξ and is isometric to a unit sphere in a $(2m+2)$-dimensional Euclidean space. Furthermore, one has the following properties:*

(a) *The existence of U is determined by an exterior differential system in involution.*
(b) *Any U is an E.C. vector field and defines an infinitesimal quasi-conformal contact transformation of Ω*
(c) *The vector field fU is self-conformal of conformal weight $\frac{2m+1}{m}$*

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