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RIEMANNIAN FOLIATIONS AND $\mathcal{F}$-JACOBI FIELDS

HOBUM KIM

In this report, given a Riemannian foliation $\mathcal{F}$ on a Riemannian manifold, we introduce the concept of $\mathcal{F}$-Jacobi fields along normal geodesics to investigate geometric properties of the leaves of $\mathcal{F}$.

1. Introduction

Given a Riemannian manifold (M, g_M), a Riemannian foliation $\mathcal{F}$ on M with bundle-like metric g_M is a foliation whose leaves are locally given as fibers of Riemannian submersions. Such a foliation $\mathcal{F}$ is characterized by the property that geodesics in (M, g_M) orthogonal to a leaf of $\mathcal{F}$ at one point are orthogonal to the leaves of $\mathcal{F}$ everywhere.

In this report, we introduce the concept of $\mathcal{F}$-Jacobi fields along geodesics orthogonal to the leaves of $\mathcal{F}$ which is expected to play a useful role in the study of geometric properties of the leaves of $\mathcal{F}$ in a similar way that Jacobi fields along geodesics in Riemannian manifolds play a crucial role in Riemannian geometry.

As applications, for a Riemannian foliation $\mathcal{F}$ we characterize the harmonicity and total geodesity in terms related with $\mathcal{F}$-Jacobi fields, and get an upper bound for the order of a focal point of an arbitrary leaf.

2. $\mathcal{F}$-Jacobi fields and $\mathcal{F}$-Jacobi tensors

Let $\mathcal{F}$ be a foliation on a Riemannian manifold (M, g_M). We denote the tangent bundle of M by TM, and the tangent and normal bundle of $\mathcal{F}$ by L and $L^{\perp}$, respectively. For $m \in M$, let L_m and $L_m^{\perp}$ be the fibers of L and $L^{\perp}$ through m, respectively. Let $\pi : TM \to L^{\perp}$ and $\pi^{\perp} : TM \to L$ be the canonical projections. The second fundamental form of $\mathcal{F}$ is denoted by α. For $z \in L_m^{\perp}$, the Weingarten map $W(z) : L_m \to L_m$ with respect to z is given by $g_{L^{\perp}}(\alpha(x, y), z) = g_L(W(z)x, y)$ for $x, y \in L_m$,

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where g_L and $g_{L^\perp}$ are the canonical metrics on L and $L^\perp$ induced from g, respectively. The O'neill integrability tensor is a tensor A of type (1,2) given by

$$A_XY = \pi^\perp(\nabla^M_{\pi X}\pi Y) + \pi(\nabla^M_{\pi X}\pi^\perp Y)$$

for $X, Y \in \Gamma TM$, where ∇^M is the Levi-Civita connection associated with g_M and ΓTM is the space of smooth sections of TM. Let $\gamma : I \to M$ be a unit speed geodesic orthogonal to a leaf of $\mathcal{F}$, where I is an open interval containing 0. The pullback bundle by γ of the bundle $\bigotimes^r_s L$ of tensors of type (r, s) over L will be denoted by $\gamma^* \bigotimes^r_s L$. A *tensor field over L along* γ of type (r, s) is a smooth section of $\gamma^* \bigotimes^r_s L$. In particular, a smooth section of $\gamma^* \bigotimes^1_0 L = L$ is called a vector field over L along γ. The connection ∇^L on L defined by $\nabla^L_Y X = \pi^\perp(\nabla^M_Y X)$ for $X \in \Gamma L$ and $Y \in \Gamma TM$, where ΓL is the space of smooth sections of L, is metric with respect to g_L. Let X be a vector field over L along γ. Then $\nabla^L_{\dot\gamma}\bar{X}$, where $\dot\gamma = \frac{d}{dt}\gamma$ and $\bar{X}$ is an extension of X to a section of L on a neighborhood of $\gamma(t)$, is called the covariant derivative of X along γ with respect to ∇^L and is denoted by $\frac{\nabla^L}{dt}X$. For a tensor field B over L along γ of type (r, s), the *covariant derivative* $\frac{\nabla^L}{dt}B$ *of* B *along* γ is given by

$$\begin{aligned}(\frac{\nabla^L}{dt}B)(w^1,\dots,w^r,X_1,\dots,X_s) &= \frac{d}{dt}B(w^1,\dots,w^r,X_1,\dots,X_s)\\ &-\sum_{i=1}^r B(w^1,\dots,\frac{\nabla^L}{dt}w^i,\dots,w^r,X_1,\dots,X_s)\\ &-\sum_{j=1}^s B(w^1,\dots,w^r,X_1,\dots,\frac{\nabla^L}{dt}X_j,\dots,X_s)\end{aligned}$$

for smooth sections $w^1, \dots, w^r$ of $\gamma^* \bigotimes^0_1 L$ and $X_1, \dots, X_s$ of $\gamma^* L$. We say that a tensor field B over L along γ is ∇^L-parallel along γ if $\frac{\nabla^L}{dt}B = 0$. Now, we are in a position to define $\mathcal{F}$-Jacobi fields and $\mathcal{F}$-Jacobi tensors.

DEFINITION 2.1. A vector field Y over L along γ is called an *$\mathcal{F}$-Jacobi field* if it satisfies the ordinary differential equation of second order:

$$\pi^{\perp}\left[\left(\frac{\nabla^M}{dt}\right)^2 Y + R^M(Y,\dot{\gamma})\dot{\gamma}\right] = 0, \tag{2.2}$$

or equivalently,

$$\left(\frac{\nabla^L}{dt}\right)^2 Y + \bar{R}_{\dot{\gamma}}(Y) + (A_{\dot{\gamma}})^2 Y = 0, \tag{2.3}$$

where $\bar{R}_{\dot{\gamma}}$ is the endomorphism on L_γ defined by $\bar{R}_{\dot{\gamma}}(Y) = \pi^{\perp} R^M(Y,\dot{\gamma})\dot{\gamma}$ and A is the integrability tensor.

DEFINITION 2.4. A tensor field D over L along γ of type (1,1) is called an *$\mathcal{F}$-Jacobi tensor* if $D(t)V(t)$ is an $\mathcal{F}$-Jacobi field along γ for any ∇^L-parallel vector field V over L along γ.

REMARK 2.5. D is an $\mathcal{F}$-Jacobi tensor if and only if it satisfies the equation:

$$\left(\frac{\nabla^L}{dt}\right)^2 D(t) + \bar{R}_{\dot{\gamma}} D(t) + (A_{\dot{\gamma}})^2 D(t) = 0. \tag{2.6}$$

LEMMA 2.7. *The Weingarten map $W(\dot{\gamma})$ satisfies the equation of Riccati type:*

$$\frac{\nabla^L}{dt} W(\dot{\gamma}) = W(\dot{\gamma})^2 + \bar{R}_{\dot{\gamma}} + (A_{\dot{\gamma}})^2 \tag{2.8}$$

For the proof, we refer to [5].

THEOREM 2.9. *Let J be a tensor field of type* (1,1) *over L along γ satisfying $\frac{\nabla^L}{dt} J = -W(\dot{\gamma})J$* (2.10). *Then J is an $\mathcal{F}$-Jacobi tensor.*

Proof. Let $\{E_i(t) : i = 1, \dots, p\}$ be a ∇^L- parallel orthonormal frame field of $\gamma^* L$, and let B_J and B_W be the matrices of J and $W(\dot{\gamma})$ with respect to $\{E_i : i = 1, \dots, p\}$, respectively. Then the equation (2.10) is equivalent to the matrix differential equation $\frac{d}{dt} B_J = -B_W B_J$ (2.11). It follows that the solutions of (2.10) exist, are invertible and satisfy $W(\dot{\gamma}) = -(\frac{\nabla^L}{dt} J)J^{-1}$ (2.12). Substituting (2.12) into (2.8), we find that J satisfies the equation (2.6).

3. Harmonicity and total geodesity

In this section, we characterize the harmonicity and total geodesity in terms of $\mathcal{F}$-Jacobi fields and $\mathcal{F}$-Jacobi tensors. A foliation $\mathcal{F}$ is *totally geodesic* if each of its leaves is a totally geodesic submanifold. $\mathcal{F}$ is *harmonic* if each of its leaves is a minimal submanifold. Hence $\mathcal{F}$ is totally geodesic if $W(z)$ vanishes for each $z \in L^{\perp}$, and $\mathcal{F}$ is harmonic if trace $W(z) = 0$ for each $z \in L^{\perp}$.

LEMMA 3.1. *The following are equivalent:*
(1) *Any ∇^L-parallel vector field over L along γ is an $\mathcal{F}$-Jacobi field.*
(2) *The Weingarten map $W(\dot\gamma)$ satisfies*

$$\frac{\nabla^L}{dt}W(\dot\gamma) = W(\dot\gamma)^2. \tag{3.2}$$

Proof. (1) $\Rightarrow$ (2): By hypothesis, the identity tensor field $I(t)$ over L along γ is an $\mathcal{F}$-Jacobi tensor. Therefore, we have

$$\left(\frac{\nabla^L}{dt}\right)^2 I(t) + \bar{R}_{\dot\gamma} + (A_{\dot\gamma})^2 = \bar{R}_{\dot\gamma} + (A_{\dot\gamma})^2 = 0.$$

Thus (3.2) follows from (2.8). The converse is obvious.

LEMMA 3.3. (1) *If $\mathcal{F}$ is totally geodesic, then every ∇^L-parallel vector field over L along γ is an $\mathcal{F}$-Jacobi field.*

(2) *Conversely, suppose that every maximal geodesic γ orthogonal to the leaves of $\mathcal{F}$ has at least one point where the Weingarten map $W(\dot\gamma)$ vanishes and that every ∇^L-parallel vector field over L along γ is an $\mathcal{F}$-Jacobi field. Then $\mathcal{F}$ is totally geodesic.*

Proof. (1) is clear. To prove (2), we note that $W(\dot\gamma)$ satisfies the equation $\frac{\nabla^L}{dt}W(\dot\gamma) = W(\dot\gamma)^2$ and vanishes for some t. Hence $W(\dot\gamma)$ vanishes identically along γ.

THEOREM 3.4. *Let $\mathcal{F}$ be a Riemannian foliation on a complete Riemannian manifold M with bundle-like metric g_M. Then $\mathcal{F}$ is totally geodesic if and only if any ∇^L-parallel vector field over L along any geodesic orthogonal to the leaves of $\mathcal{F}$ is an $\mathcal{F}$-Jacobi field.*

Proof. Sufficiency is clear. To prove the necessity, choose a ∇^L-parallel orthonormal frame $\{E_1, \dots , E_p\}$ of L along a maximal geodesic

γ orthogonal to the leaves of $\mathcal{F}$. Since M is complete, γ is defined on the whole real line. Let $[W_{ij}(t)]$ be the matrix of $W(\dot{\gamma})$ with respect to the basis $\{E_1(t), \dots, E_p(t)\}$. It suffices to show that $W_{ij}(0) = 0$ for all $i, j = 1, \dots, p$. We may assume that $[W_{ij}(0)]$ is a diagonal matrix. Suppose $W_{ii}(0) > 0$ for some $1 \le i \le p$. From (3.2), we have $\frac{d}{dt}W_{ii} = (W_{ii})^2 + \sum_{k \neq i} W_{kk}^2$. Since $\sum_{k \neq i} W_{kk}^2 \ge 0$, $W_{ii}(t)$ increases not slower than the solution of the equation $\frac{d}{dt}Q = Q^2$, with the initial condition $Q(0) = W_{ii}(0)$ (3.4). But the solution of the initial value problem (3.4) is given by $Q(t) = \frac{W_{ii}(0)}{1 - tW_{ii}(0)}$ which blows up at $t = \frac{1}{W_{ii}(0)}$. Hence $W_{ii}(t)$ should also blow up at some $0 \le t < \infty$, contradicting the fact that $[W_{ij}]$ is a global solution. In the case $W_{ii}(0) < 0$, taking the reverse geodesic, we also get a contradiction by the same way as above.

Harmonic foliation can be characterized in terms of $\mathcal{F}$-Jacobi as follows.

THEOREM 3.5. *$\mathcal{F}$ is harmonic if and only if $\frac{d}{dt}(detJ)$ vanishes identically along any geodesic orthogonal to the leaves of $\mathcal{F}$.*

Proof. It suffices to show that Trace $W(\dot{\gamma}) = -(\frac{d}{dt}detJ)/detJ$. Let $\{E_1, \dots, E_p\}$ be a ∇^L-parallel frame field of L along γ, and let $Y_i = JE_i$. Then we have $(detJ)E_1 \wedge \cdots \wedge E_p = Y_1 \wedge \cdots \wedge Y_p$. Taking $\frac{\nabla^L}{dt}$ on both sides of the above equation, we get

$$(3.6) \qquad (\frac{d}{dt}detJ)E_1 \wedge \cdots \wedge E_p = \sum_{i=1}^{p} Y_1 \wedge \cdots \wedge \frac{\nabla^L}{dt}Y_i \wedge \cdots \wedge Y_p.$$

Since $\frac{\nabla^L}{dt}J = -W(\dot{\gamma})J$, we have

$$\frac{\nabla^L}{dt}Y_i = \frac{\nabla^L}{dt}(JE_i) = (\frac{\nabla^L}{dt}J)E_i = -W(\dot{\gamma})JE_i = -W(\dot{\gamma})Y_i.$$

It follows that

$$\begin{aligned} &\sum_{i=1}^{p} Y_1 \wedge \cdots \wedge \frac{\nabla^L}{dt}Y_i \wedge \cdots \wedge Y_p \\ =&\sum_{i=1}^{p} Y_1 \wedge \cdots \wedge (-W(\dot{\gamma})Y_i) \wedge \cdots \wedge Y_p \\ =& -(detJ)(TraceW(\dot{\gamma}))E_1 \wedge \cdots \wedge E_p. \end{aligned}$$

The above together with (3.6) completes the proof.

4. Focal points

In this section, we get an upper bound for the order of a focal point of a leaf of a Riemannian foliation. Let S be a submanifold of a Riemannian manifold M and $N(S)$ its normal bundle. The restriction of the exponential map of M on $N(S)$ gives the map $\exp : N(S) \to M$. For $x \in S$, let $N(S)(x)$ be the fiber of $N(S)$ over x. $v \in N(S)(x)$ is called a *focal point* of S if $d\exp$ is singular at v. If ρ is a ray from 0 to v, then $\exp v$ is called a *focal point* of S *along* $\exp \circ \rho$. The *order* of a focal point is the dimension of the linear space annihilated by $d\exp$. Let γ be a geodesic segment orthogonal to S at $\gamma(0)$. A Jacobi field Y along γ is called an *S-Jacobi field* if Y is perpendicular to γ, $Y(0) \in T_{\gamma(0)}S$ and $W(\dot\gamma(0))Y(0) + \nabla^M_{\dot\gamma(0)}Y$ is perpendicular to $T_{\gamma(0)}S$. It is well-known that S-Jacobi fields form a linear space of dimension $n-1$, where $n = \dim M$. It is shown in 11.2 and 11.3 of [2] that $\gamma(b)$ is a focal point of order r of S along γ if and only if there are r-linearly independent S-Jacobi fields which vanish at b.

Now, we restrict our attention to the leaves of a Riemannian foliation $\mathcal{F}$ on a Riemannian manifold M with bundle-like metric g_M. For $m \in M$ let $\mathcal{L}_m$ denote the leaf of $\mathcal{F}$ through m, and let γ be a geodesic orthogonal to the leaves of $\mathcal{F}$ defined on an open interval containing 0.

LEMMA 4.1. *Suppose the orthogonal complement of $\mathcal{F}$ is involutive. Then,*

(1) *for any $\mathcal{L}_{\gamma(0)}$-Jacobi field $Y_1, Y = \pi^\perp Y_1$ is an $\mathcal{F}$-Jacobi field satisfying the initial condition $(\frac{\nabla^L}{dt}Y)(0) = -W(\dot\gamma(0))Y(0)$.*

(2) *Conversely, if Y is an $\mathcal{F}$-Jacobi field given by $Y = JE$, where J is a solution of $\frac{\nabla^L}{dt}J = -W(\dot\gamma)J$ and E is any ∇^L-parallel vector field over L along γ, then Y is an $\mathcal{L}_{\gamma(0)}$-Jacobi field.*

Proof. Since $\mathcal{F}$ is Riemannian and $F^\perp$ involutive, it follows that $\mathcal{F}^\perp$ is totally geodesic. Thus for any $U \in \Gamma L$ and $Z_1, Z_2 \in \Gamma L^\perp$, we have $\nabla^M_{Z_1}U \in \Gamma L$ and $\nabla^M_{Z_1}Z_2 \in \Gamma L^\perp$. Moreover, $R(u,v)w \in L_m$ for $u \in L_m$ and $v, w \in L^\perp_m$, $m \in M$. Decomposing Y_1 as $Y_1 = \pi^\perp Y_1 + \pi Y_1$, we easily see that $Y = \pi^\perp Y_1$ satisfies the equation for $\mathcal{F}$-Jacobi fields. Moreover, from the condition that $W(\dot\gamma(0))Y_1(0) + \nabla^M_{\dot\gamma(0)}Y_1$ is perpendicular to $L_{\gamma(0)}$, we have $W(\dot\gamma(0))Y(0) + (\frac{\nabla^L}{dt}Y)(0) = 0$. This completes the proof of (1). To prove (2), we note that Y satisfies the equation for Jacobi fields. Moreover, we have $\frac{\nabla^M}{dt}Y = \frac{\nabla^L}{dt}Y = (\frac{\nabla^L}{dt}J)E = -W(\dot\gamma)JE = -W(\dot\gamma)Y$. Hence, we have $W(\dot\gamma(0))Y(0) + \nabla^M_{\dot\gamma(0)}Y = 0$ and the proof of (2) is complete.

THEOREM 4.2. *Let $\gamma(t_0)$ be a focal point of $\mathcal{L}_{\gamma(0)}$ along γ. If $\mathcal{F}^\perp$ is involutive, then order of $\gamma(t_0) \leq q-1$.*

Proof. By (2) of Lemma 4.1, there are at least p linearly independent nowhere vanishing $\mathcal{L}_{\gamma(0)}$-Jacobi fields along γ. But the dimension of the space of all $\mathcal{L}_{\gamma(0)}$-Jacobi fields is $n-1$. Hence it follows that order of $\gamma(t_0)$ along $\gamma \leq n-1-p = q-1$.

COROLLARY 4.3. *If $\mathcal{F}$ is a Riemannian foliation of codimension one, then no leaf of $\mathcal{F}$ has focal points.*

References

1. K. Abe, *Application of a Riccati type differential equation to Riemannian manifolds with totally geodesic distributions*, Tôhoku Math. J. **25** (1973), 425–444.
2. R. L. Bishop and R. J. Crittenden, *Geometry of Manifolds*, Academic Press, New York, 1964.
3. J. Eschenburg and J. J. O'Sullivan, *Growth of Jacobi Fields and Divergence of Geodesics*, Math. Z. **150** (1976), 221–237.
4. F. W. Kamber and Ph. Tondeur, *Harmonic foliations*, Lecture Notes in Math., vol. 949, Springer, Berlin-New York, 1982, 87–121.
5. H. Kim and Ph. Tondeur, *Riemannian foliations on manifolds with non-negative curvature*, Manuscripta Math. **74** (1992), 39–45.
6. B. O'neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.
7. Ph. Tondeur, *Foliations on Riemannian manifolds*, Universitext, Springer, New York, 1988.
8. S. Yorozu, *Behavior of geodesics in foliated manifolds with bundle-like metrics*, J. Math. Soc. Japan **35** (1983), 251–272.

Department of Mathematics
Yonsei University
Seoul 120-749, Korea