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ON REAL HYPERSURFACES OF TYPE A IN A COMPLEX SPACE FORM (II)

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§1. Introduction

A complex n-dimensional Kähler manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. A complete and simply connected complex space form consists of a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n\mathbb{C}$, according as $c > 0, c = 0$ or $c < 0$.

In his study of real hypersurfaces of $P_n\mathbb{C}$, Takagi [12] classified all homogeneous real hypersurfaces and Cecil and Ryan [3] showed also that they are realized as the tubes of constant radius over Kähler submanifolds if the structure vector field is principal. Real hypersurfaces of $H_n\mathbb{C}$ have also investigated by Berndt [2], Montiel [9], Montiel and Romero [10] and so on, and Berndt [2] classified all homogeneous real hypersurfaces of $H_n\mathbb{C}$ and showed that they are realized as the tubes of constant radius over certain submanifolds. According to Takagi's classification theorem and Berndt's one, the principal curvatures and their multiplicities of homogeneous real hypersurfaces of $M_n(c)$ are given.

Now, let M be a real hypersurface of $M_n(c), c \neq 0$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler metric and the almost complex structure of $M_n(c)$. We denote by A the shape operator in the direction of the unit normal and by ∇ the Riemannian connection on M. Then Okumura [11] and Montiel and Romero [10] proved the following

THEOREM A. *Let M be a real hypersurface of $P_n\mathbb{C}, n \geqq 2$. If it satisfies*

$$A\phi - \phi A = 0, \tag{1.1}$$

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then M is locally a tube of radius r over one of the following Kähler submanifolds:

(A_1) *a hyperplane $P_{n-1}\mathbb{C}$, where $0 < r < \pi/2$,*
(A_2) *a totally geodesic $P_k\mathbb{C}$ $(1 \leqq k \leqq n-2)$, where $0 < r < \pi/2$.*

THEOREM B. *Let M be a real hypersurface of $H_n\mathbb{C}, n \geqq 2$. If it satisfies* (1.1), *then M is locally one of the following hypersurfaces:*

(A_0) *a horosphere in $H_n\mathbb{C}$, i.e., a Montiel tube,*
(A_1) *a tube of a totally geodesic hyperplane $H_{n-1}\mathbb{C}$,*
(A_2) *a tube of a totally geodesic $H_k\mathbb{C}$ $(1 \leqq k \leqq n-2)$.*

Such real hypersurfaces in Theorems A and B are said to be of *type A*. On the other hand, Ki, Kim and Lee [4] gave the following

THEOREM C. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 2$. If it satisfies*

$$\nabla_\xi A = 0, \quad g(A\xi, \xi) \neq 0, \tag{1.2}$$

then M is of type A.

Let T_0 be a distribution defined by the subspace $T_0(x) = \{u \in T_xM : g(u, \xi(x)) = 0\}$ of the tangent space T_xM of M at any point x, which is called the *holomorphic distribution.* As an example of non-homogeneous real hypersurfaces in $M_n(c), c \neq 0$, we have ruled real ones. It is also seen in Kimura [6] and Ahn, Lee and Suh [1] that ruled real hypersurfaces are characterized by the holomorphic distribution T_0. On the other hand, for the Hopf fibration $\pi : S^{2n+1}(1) \to P_n\mathbb{C}$, the projection of a hypersurface with parallel second fundamental form in $S^{2n+1}(1)$ becomes a real one in $P_n\mathbb{C}$, which satisfies $\nabla_\xi A = 0$. Thus it seems to be interesting the property for $\nabla_\xi A$ restricted to T_0.

The purpose of this article is to prove the following generalized properties of Theorem C.

THEOREM 1. *Let M be a real hypersurface of $P_n\mathbb{C}, n \geqq 3$. If $\nabla_\xi A(X) = 0$ for any vector field X in T_0, then M is locally congruent to one of the following:*

(a) *a non-homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a certain Kähler submanifold in $P_n\mathbb{C}$,*

(b) *a real hypersurface of type A.*

THEOREM 2. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$. If it satisfies*

$$\nabla_\xi A(X) = 0, \quad g(A\xi, \xi) \neq 0 \tag{1.3}$$

for any vector field X in T_0, then M is of type A.

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§2. Preliminaries

First of all, we recall fundamental properties of real hypersurfaces of a complex space form. Let M be a real hypersurface of a complex n-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature c, and let C be a unit normal vector field on a neighborhood in M. We denote by J the almost complex structure of $M_n(c)$. For a local vector field X on the neighborhood in M, the images of X and C under the linear transformation J can be represented as

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on the neighborhood in M, respectively. Then it is seen that $g(\xi, X) = \eta(X)$, where g denotes the Riemannian metric tensor on M induced from the metric tensor on $M_n(c)$. The set of tensors (ϕ, ξ, η, g) is called an *almost contact metric structure* on M. They satisfy the following properties:

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. Furthermore, the covariant derivatives of the structure tensors are given by

$$\nabla_X \xi = \phi AX, \quad \nabla_X \phi(Y) = \eta(Y)AX - g(AX, Y)\xi \tag{2.1}$$

for any vector fields X and Y on M, where ∇ is the Riemannian connection on M and A is the shape operator of M in the direction of C.

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively obtained:

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y \\ &+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\} \\ &+g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.2}$$

$$\nabla_X A(Y) - \nabla_Y A(X) = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\}, \tag{2.3}$$

where R denotes the Riemannian curvature tensor of M.

§3. Proof of Theorems

Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$ and assume that

$$\nabla_\xi A(X) = 0, \quad X \in T_0. \tag{3.1}$$

By the assistance of (2.3), it turns out to be

$$\nabla_Y A(\xi) = -\frac{c}{4}\phi Y, \quad Y \in T_0. \tag{3.2}$$

Differentiating this equation with respect to X covariantly and taking account of (2.1), we get

$$\nabla_X\nabla_Y A(\xi) + \nabla_{\nabla_X Y}A(\xi) + \nabla_Y A(\phi AX) = \frac{c}{4}\{g(AX,Y)\xi - \phi\nabla_X Y\}$$

for any vector fields X and Y in T_0. Since the component of the vector $\nabla_X Y$ in the direction of ξ is given by $-g(\phi AX, Y)$ by the first equation of (2.1), we have the following orthogonal decomposition

$$\nabla_X Y = (\nabla_X Y)_0 - g(\phi AX, Y)\xi,$$

where $(\nabla_X Y)_0$ denotes the T_0-component of $\nabla_X Y$. By (2.3), (3.1) and the covariant differentiation which is given above and the orthogonal decomposition, we get

$$\nabla_X \nabla_Y A(\xi) = g(\phi AX, Y)\nabla_\xi A(\xi) - \nabla_Y A(\phi AX) + \frac{c}{4} g(AX, Y)\xi \tag{3.3}$$

for any vector fields X and Y in T_0. As is well known, the Ricci formula for the shape operator A is given by

$$\nabla_X \nabla_Y A(Z) - \nabla_Y \nabla_X A(Z) = R(X, Y)(AZ) - A(R(X, Y)Z)$$

for any vector fields X, Y and Z. Accordingly, by putting $Z = \xi$ and taking X and Y in the distribution T_0, we obtain by the Gauss equation (2.2) and (3.3) yield

$$\begin{aligned} g((A\phi + \phi A)X, Y)&\nabla_\xi A(\xi) + \nabla_X A(\phi AY) - \nabla_Y A(\phi AX) \\ &= \frac{c}{4}\{g(Y, A\xi)X - g(X, A\xi)Y \\ &\quad + g(\phi Y, A\xi)\phi X - g(\phi X, A\xi)\phi Y - 2g(\phi X, Y)\phi A\xi\} \\ &\quad -g(Y, A\xi)A^2X + g(X, A\xi)A^2Y + g(Y, A^2\xi)AX \\ &\quad - g(X, A^2\xi)AY. \end{aligned} \tag{3.4}$$

Now, we consider first the case where the structure vector field ξ is principal. Then it is easily seen that $\nabla_\xi A = 0$ holds under the assumption (3.1). So, suppose that the structure vector field ξ is not principal. Then we can put $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in the holomorphic distribution T_0, and α and β are smooth functions on M. We may consider that the function β does not vanish identically on M. Let M_0 be the non-empty open subset of M consisting of points x at which $\beta(x) \neq 0$. Hereafter unless otherwise stated, we shall discuss on the subset M_0 of M. For the object, we shall express (3.4) with the simpler form. By the form $A\xi = \alpha\xi + \beta U$, the equation (3.4) can be reformed as

$$\begin{aligned} g((A\phi+\phi A)X, Y)&\nabla_\xi A(\xi) + \nabla_X A(\phi AY) - \nabla_Y A(\phi AX) \\ &= \frac{c}{4}\beta\{g(Y, U)X - g(X, U)Y - g(Y, \phi U)\phi X + g(X, \phi U)\phi Y \\ &\quad - 2g(\phi X, Y)\phi U\} + \beta[-g(Y, U)A^2X + g(X, U)A^2Y \\ &\quad +\{\alpha g(Y, U) + g(Y, AU)\}AX \\ &\quad -\{\alpha g(X, U) + g(X, AU)\}AY] \end{aligned} \tag{3.5}$$

for any vector fields X and Y in T_0. Differentiating $A\xi = \alpha\xi + \beta U$ with respect to ξ covariantly, we have by (2.1)

$$\nabla_\xi A(\xi) = d\alpha(\xi)\xi + d\beta(\xi)U + \alpha\beta\phi U - \beta A\phi U + \beta\nabla_\xi U. \tag{3.6}$$

Since it is easily seen by (2.1) and by the choice of the vector field U that the vectors $A\phi U$ and $\nabla_\xi U$ are both orthogonal to ξ, we see

$$g(\nabla_\xi A(\xi), \xi) = d\alpha(\xi),$$

i.e., we have

$$\nabla_\xi A(\xi) = d\alpha(\xi)\xi, \tag{3.7}$$

because $\nabla_\xi A(\xi)$ is orthogonal to T_0 by (3.1). On the other hand, taking X and Y in the distribution T_0, we have

$$g(\nabla_X A(\phi AY), \xi) = -\frac{c}{4} g(AX, Y), \tag{3.8}$$

where we have used (3.2). By taking account of (3.7) and (3.8), the inner product of (3.5) and ξ gives us

$$\begin{aligned} d\alpha(\xi)&g((A\phi + \phi A)X, Y) \\ &= 2\beta^2\{g(X,U)g(Y,AU) - g(Y,U)g(X,AU)\} \end{aligned} \tag{3.9}$$

for any vector fields X and Y in T_0. Hence we get by the above equation

$$\begin{aligned} d\alpha(\xi)(A\phi + \phi A)X =& 2\beta^2\{g(X,U)AU - g(AX,U)U\} \\ &- \beta\{2\beta^2 g(X,U) + d\alpha(\xi)g(X,\phi U)\}\xi \end{aligned} \tag{3.10}$$

for any vector filed X in T_0.

Now, we can consider that there is a vector field V in the holomorphic distribution T_0 in such a way that AU is expressed as a linear combination of the vector fields ξ, U and V, where U and V are orthonormal. Namely, since the shape operator A is symmetric, we may put $AU = \beta\xi + \gamma U + \delta V$, where γ and δ are smooth functions on M_0. Putting $X = U$ in (3.10) and using the expression of AU, we get

$$d\alpha(\xi)A\phi U = 2\beta^2\delta V - \gamma d\alpha(\xi)\phi U - \delta d\alpha(\xi)\phi V. \tag{3.11}$$

Consequently, acting the linear transformation ϕ to the above equation, we have

$$d\alpha(\xi)\phi A\phi U = \gamma d\alpha(\xi)U + \delta d\alpha(\xi)V + 2\beta^2\delta\phi V. \tag{3.12}$$

Putting $X = \phi U$ in (3.10) again and making use of the decomposition of AU, we get

$$d\alpha(\xi)\phi A\phi U = \{\gamma d\alpha(\xi) - 2\beta^2\delta g(\phi U, V)\}U + \delta d\alpha(\xi)V,$$

from which together with (3.12), it follows that we have

$$\beta^2\delta\{\phi V + g(\phi U, V)U\} = 0.$$

Let M_1 be the open subset of M_0 consisting of points x at which $\delta(x) \neq 0$. Suppose that M_1 is not empty. On M_1, we have $V = \pm\phi U$ by the above equation, because U and ϕV are both unit. Without loss of generality, we may suppose that $V = \phi U$. Thus we have

$$d\alpha(\xi)A\phi U = \delta d\alpha(\xi)U + \{2\beta^2\delta - \gamma d\alpha(\xi)\}\phi U \tag{3.13}$$

on M_1. On the other hand, by (3.11), we have $d\alpha(\xi)A\phi U = -\gamma d\alpha(\xi)\phi U$ on $M_0 - M_1$. Consequently, (3.13) holds on M_0.

Next, we investigate the mutual relations among the functions α, β,γ and δ. First we differentiate AU with respect to ξ covariantly. Then, taking account of (2.1) and (3.1), we get

$$\begin{aligned} A\nabla_\xi U = &\{d\beta(\xi) - \beta\delta\}\xi + d\gamma(\xi)U \\ &+\{\beta^2 + d\delta(\xi)\}\phi U + \gamma\nabla_\xi U + \delta\phi\nabla_\xi U \end{aligned} \tag{3.14}$$

on M_1. Furthermore, this equation holds on M_0. By the forms $A\xi = \alpha\xi + \beta U$ and $AU = \beta\xi + \gamma U + \delta\phi U$, it is easily seen that the following equations

$$\begin{aligned} &g(A\nabla_\xi U, \xi) = 0, \\ &g(A\nabla_\xi U, U) = \delta g(\nabla_\xi U, \phi U), \\ &d\alpha(\xi)g(A\nabla_\xi U, \phi U) = \{2\beta^2\delta - \gamma d\alpha(\xi)\}g(\nabla_\xi U, \phi U) \end{aligned}$$

are obtained, where we have used (3.13) to derive the last equation. Then we consider the inner product of (3.14) and ξ, U and ϕU respectively. Taking account of the above three equations, we have the following mutual relations:

$$d\beta(\xi) = \beta\delta. \tag{3.15}$$

$$d\gamma(\xi) = 2\delta g(\nabla_\xi U, \phi U), \tag{3.16}$$

$$2\{\beta^2\delta - \gamma d\alpha(\xi)\}g(\nabla_\xi U, \phi U) = d\alpha(\xi)\{\beta^2 + d\delta(\xi)\}. \tag{3.17}$$

On the other hand, we take here the inner product of (3.6) and ϕU. Then the inner product of the left hand side vanishes identically by (3.1) and therefore it implies

$$d\alpha(\xi)g(\nabla_\xi U, \phi U) = 2\beta^2\delta - (\alpha + \gamma)d\alpha(\xi), \tag{3.18}$$

where we have used (3.13). Also, by taking account of (3.10) and the form AU, it is easily seen that we have

$$\begin{aligned} &d\alpha(\xi)(A\phi + \phi A)X \\ &\quad = -\beta d\alpha(\xi)g(X, \phi U)\xi - 2\beta^2\delta g(X, \phi U)U + 2\beta^2\delta g(X, U)\phi U \end{aligned} \tag{3.19}$$

for any vector field X in T_0.

Now, let $L(\xi, U, \phi U)$ be a distribution defined by the subspace $L_x(\xi, U, \phi U)$ in the tangent space T_xM spanned by the vectors $\xi(x)$, $U(x)$ and $\phi U(x)$ at any point x in M_0. Then the subspace $L_x(\xi, U, \phi U)$ is A-invariant by (3.13) and also ϕ-invariant. Let T_1 be the orthogonal complement in the tangent bundle TM of the distribution $L(\xi, U, \phi U)$. Since the distribution $L(\xi, U, \phi U)$ is A-invariant, the orthogonal distribution T_1 is also A-invariant and moreover it is ϕ-invariant, too. Let M_2 be the subset of M_0 consisting of points x at which $d\alpha(\xi)(x) \neq 0$. If the subset M_2 is empty, then we can derive that $\nabla_\xi A = 0$ from the assumption (3.1) and (3.7). So, suppose that M_2 is not empty. Hereafter unless otherwise stated, our discussion will be continued on M_2. Accordingly, by (3.19), we have

$$(A\phi + \phi A)X = 0, \quad X \in T_1.$$

By differentiating this equation with respect to ξ covariantly and combining with (2.1) and (3.1), it implies that

$$(A\phi + \phi A)\nabla_\xi X = 0, \quad X \in T_1,$$

because T_1 is A-invariant and ϕ-invariant. The inner product of this equation and ξ yields $g(\nabla_\xi U, \phi X) = 0$ by (2.1). Since T_1 is ϕ-invariant, we get

$$g(\nabla_\xi U, X) = 0, \quad X \in T_1.$$

Evidently, we get

$$g(\nabla_\xi U, \xi) = 0, \quad g(\nabla_\xi U, U) = 0,$$

from which it follows that we can express $\nabla_\xi U$ as

$$\nabla_\xi U = \varepsilon\phi U, \tag{3.20}$$

where ε is a smooth function on M_2. Accordingly the equations (3.16) $\sim$ (3.18) can be rewritten as follows:

$$d\gamma(\xi) = 2\delta\varepsilon, \tag{3.16'}$$

$$d\alpha(\xi)\{\beta^2 + 2\gamma\varepsilon + d\delta(\xi)\} = 2\varepsilon\beta^2\delta, \tag{3.17'}$$

$$d\alpha(\xi)(\alpha + \gamma + \varepsilon) = 2\beta^2\delta. \tag{3.18'}$$

By (3.17′) and (3.18′), we see

$$d\delta(\xi) = \varepsilon(\alpha - \gamma + \varepsilon) - \beta^2. \tag{3.21}$$

And, from (3.13) and (3.18′), we get also

$$A\phi U = \delta U + (\alpha + \varepsilon)\phi U. \tag{3.22}$$

On the other hand, differentiating the function $g(A\phi U, \phi U) = \alpha + \varepsilon$ with respect to ξ exteriorly and using (2.1), (3.1) and (3.20), we have

$$d(\alpha + \varepsilon)(\xi) = -2\delta\varepsilon,$$

which, by (3.16′), implies

$$d(\alpha+\gamma+\varepsilon)(\xi)=0. \tag{3.23}$$

Now, since T_1 is A-invariant, there is a principal vector field X_0 in T_1 with principal curvature λ_0, where X_0 is unit. Then it turns out to be $A\phi X_0 = -\lambda_0\phi X_0$. Differentiating $AX_0 = \lambda_0 X_0$ with respect to ξ covariantly, we have

$$A\nabla_\xi X_0 = d\lambda_0(\xi)X_0 + \lambda_0\nabla_\xi X_0$$

by (3.1), which implies by the inner product of this equation with X_0 that

$$d\lambda_0(\xi)=0. \tag{3.24}$$

On the other hand, differentiating $A\xi = \alpha\xi + \beta U$ with respect to X_0 covariantly and applying (2.1) and (3.2), we have

$$\beta\nabla_{X_0}U = -d\alpha(X_0)\xi - d\beta(X_0)U - \left(\lambda_0^2+\alpha\lambda_0+\frac{c}{4}\right)\phi X_0.$$

Since the vector field $\nabla_{X_0}U$ is orthogonal to ξ and U, it implies

$$\begin{cases} d\alpha(X_0)=0, \quad d\beta(X_0)=0, \\ \beta\nabla_{X_0}U = -\left(\lambda_0^2+\alpha\lambda_0+\frac{c}{4}\right)\phi X_0. \end{cases} \tag{3.25}$$

Furthermore, differentiating $AU = \beta\xi + \gamma U + \delta\phi U$ with respect to X_0 covariantly and making use of (2.1) and (3.25), we have

$$\begin{aligned} \beta\nabla_{X_0}A(U) =& \beta d\gamma(X_0)U + \beta d\delta(X_0)\phi U + \delta\left(\lambda_0^2+\alpha\lambda_0+\frac{c}{4}\right)X_0 \\ &+ \left\{\lambda_0\beta^2 - (\lambda_0+\gamma)\left(\lambda_0^2+\alpha\lambda_0+\frac{c}{4}\right)\right\}\phi X_0. \end{aligned}$$

Thus we get

$$\begin{cases} \beta g(\nabla_{X_0}A(U),X_0) = \delta\left(\lambda_0^2+\alpha\lambda_0+\frac{c}{4}\right), \\ \beta g(\nabla_{X_0}A(U),\phi X_0) = \lambda_0\beta^2 - (\lambda_0+\gamma)\left(\lambda_0^2+\alpha\lambda_0+\frac{c}{4}\right). \end{cases} \tag{3.26}$$

Similarly, by (3.22), we get

$$(3.27)\qquad \begin{cases} \beta g(\nabla_{X_0}A(\phi U), X_0) = (-\lambda_0 + \alpha + \varepsilon)\left(\lambda_0^2 + \alpha\lambda_0 + \frac{c}{4}\right), \\ \beta g(\nabla_{X_0}A(\phi U), \phi X_0) = -\delta\left(\lambda_0^2 + \alpha\lambda_0 + \frac{c}{4}\right). \end{cases}$$

And, we shall consider the equation (3.5) for the unit vector field X_0. Putting $Y = U$ in it, we have

$$\nabla_{X_0}A(\phi AU) - \nabla_U A(\phi AX_0) = \beta\left\{-\lambda_0^2 + (\alpha+\gamma)\lambda_0 + \frac{c}{4}\right\}X_0.$$

Taking account of (2.3), (3.26) and (3.27) and by the direct calculation, we see that any principal curvature of the shape operator $A|T_1$ restricted to T_1 satisfies the following quadratic equation:

$$(3.28)\qquad \begin{aligned} &y^4 + \alpha y^3 + \left\{\gamma(\alpha+\varepsilon) - \delta^2 + \frac{c}{4}\right\}y^2 \\ &\quad + \left\{\alpha\gamma(\alpha+\varepsilon) - \alpha\delta^2 - \beta^2(\alpha+\gamma)\right\}y \\ &\quad - \frac{c}{4}\{\beta^2 + \delta^2 - \gamma(\alpha+\varepsilon)\} \\ &= 0. \end{aligned}$$

Next, suppose that there are a principal curvature λ of $A|T_1$ and a point x in M_2 at which $\lambda(x) = 0$. For the principal curvature λ, let M_λ be the subset of M_2 consisting of points x at which $\lambda(x) = 0$. Then, by (3.28), we have

$$(3.29)\qquad \beta^2 + \delta^2 - \gamma(\alpha+\varepsilon) = 0$$

on M_λ. Suppose first that $AX = 0$ for any vector field X in T_1 on the interior Int M_λ of M_λ. Since the vector field $\nabla_X Y$ for any vector fields X and Y in T_1 is expressed as

$$\beta\nabla_X Y = \beta(\nabla_X Y)_1 + \frac{c}{4}\{g(\phi X, Y)U - g(X,Y)\phi U\}$$

by (2.1) and (3.25), we have

$$\beta A\nabla_X Y = \frac{c}{4}\{g(\phi X, Y)AU - g(X,Y)A\phi U\}$$

on Int M_λ, where $(\nabla_X Y)_1$ denotes the T_1-component of $\nabla_X Y$. Exchanging X and Y in the above equation and substituting the second one from the first one, we have

$$g(\phi X, Y)(\gamma U + \delta \phi U) = 0, \quad X, Y \in T_1$$

on Int M_λ. Hence $\gamma = \delta = 0$, a contradiction by (3.29). So, there is a principal curvature μ of $A|T_1$ such that $\mu(x) \neq 0$ at x in Int M_λ. By (3.28) and (3.29), the principal curvature is a root of the cubic equation

$$y^3 + \alpha y^2 + \left(\beta^2 + \frac{c}{4}\right) y - \beta^2 \gamma = 0.$$

Since μ and $-\mu$ are non-zero roots, we have

$$y^2 + \left(\beta^2 + \frac{c}{4}\right) = 0, \quad \alpha y^2 - \beta^2 \gamma = 0.$$

Accordingly, we get

$$\beta^2(\alpha + \gamma) + \frac{c}{4}\alpha = 0.$$

By using (3.15), (3.16′), (3.18′) and (3.23), the twice exterior differentiation of this equation with respect to ξ gives us to $\beta^2 \delta = 0$, a contradiction by (3.18′) and (3.29). Thus we see that the interior of the subset M_λ is empty, which means that any principal curvature of $A|T_1$ has no zero points almost everywhere in M_λ. On $M_2 - M_\lambda$, the principal curvature λ is a solution of the equation (3.28). Since $-\lambda (\neq 0)$ is also principal curvature, it yields

$$\alpha \lambda^2 + \left\{ \alpha \gamma(\alpha + \varepsilon) - \alpha \delta^2 - \beta^2(\alpha + \gamma) \right\} = 0. \tag{3.30}$$

Differentiating this equation with respect to ξ covariantly and taking account of (3.15), (3.16′), (3.21), (3.23), (3.24) and (3.30), we have

$$\gamma d\alpha(\xi) - 2\alpha\delta(\gamma + \varepsilon) = 0,$$

from which it follows that again by (3.18′)

$$\delta\{\beta^2 \gamma - \alpha(\gamma + \varepsilon)(\alpha + \gamma + \varepsilon)\} = 0 \tag{3.31}$$

on $M_2 - M_\lambda$. Here we consider the subset $M_2 - M_1$ which consists of points x at which $\beta(x) \neq 0$, $d\alpha(\xi)(x) \neq 0$ and $\delta(x) = 0$. Suppose that the interior of the subset $M_2 - M_1$ is not empty. Then, by (3.15), (3.16′), (3.17′) and (3.18′), we have

$$d\beta(\xi) = 0, \qquad d\gamma(\xi) = 0,$$
$$\beta^2 + 2\gamma\varepsilon = 0, \quad \alpha + \gamma + \varepsilon = 0$$

on the interior of $M_2 - M_1$. By these equations, we get $\gamma d\alpha(\xi) = 0$. Hence we obtain $\beta = 0$, a contradiction. This means that the interior of $M_2 - M_1$ must be empty. Consequently, any point x almost everywhere in M_2 satisfies $\delta(x) \neq 0$. Thus, by continuity, we have by (3.31)

$$\beta^2\gamma - \alpha(\gamma + \varepsilon)(\alpha + \gamma + \varepsilon) = 0$$

on $M_2 - M_\lambda$. Again, differentiating this equation with respect to ξ covariantly and taking account of (3.15), (3.16′), (3.18′) and (3.23), we have $\alpha = 0$. Hence we get

$$d\alpha(\xi) = 0$$

on $M_2 - M_\lambda$, namely, by continuity, on M_2. This means that M_2 is empty. Consequently, we have the following proposition. This shows that Theorems 1 and 2 are verified by a theorem due to Kimura and Maeda [7] and Theorem C, respectively.

PROPOSITION 3.1. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$. If it satisfies*

$$\nabla_\xi A(X) = 0, \quad X \in T_0,$$

then $\nabla_\xi A = 0$.

As a direct consequence of Theorem 2, we find the following corollaries.

COROLLARY 3.2. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$. If the structure vector field ξ is principal and if it satisfies*

$$g(\nabla_\xi A(X), Y) = 0, \quad g(A\xi, \xi) \neq 0$$

for any vector fields X and Y in T_0, then M is of type A.

Proof. By the assumption, we see $A\xi = \alpha\xi$ and then α is constant.(See [5] and [8].) Accordingly we have

$$\nabla_\xi A(\xi) + A\nabla_\xi \xi = \alpha \nabla_\xi \xi,$$

which implies $\nabla_\xi A(\xi) = 0$, where we have used that $\nabla_\xi \xi = 0$. Hence, by the assumption, we get $\nabla_\xi A(X) = 0$ for any vector field X in T_0. This completes the proof.

Now, let $\mathcal{L}_\xi$ be the Lie derivative with respect to ξ. We define the second fundamental form h by $h(X,Y) = g(AX,Y)$ for any vector fields X and Y.

COROLLARY 3.3. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$. If it satisfies*

$$\mathcal{L}_\xi h(X,Y) = 0, \quad g(A\xi,\xi) \neq 0$$

for any vector field X in T_0 and any vector field Y, then M is of type A.

Proof. Because of

$$\mathcal{L}_\xi h(X,Y) = g(\nabla_\xi A(X), Y)$$

for any vector fields X and Y, the proof completes.

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