Comm. Korean Math. Soc. 9 (1994), No. 2, pp. 355–359

A BIFURCATION ANALYSIS FOR RADIALLY SYMMETRIC ENERGY MINIMIZING MAPS ON ANNULUS

DONG PYO CHI AND GIE HYUN PARK

It would be interesting to know if energy minimizing harmonic maps between manifolds have symmetric properties when the manifolds under consideration have some. In this paper, we consider among others radial symmetry. A *radially symmetric* manifold M of dimension m is the one with a point, called a *pole*, and an $O(m)$ action as an isometric rotation with respect to the pole, or more precisely a radially symmetric manifold M has a coordinate on which the metric is of the form ${ds_M}^2 = dr^2 + m(r)^2 d\theta^2$ for some function $m(r)$ depending only on r. Of course $m(0) = 0$, $m'(0) = 1$, and when $m(r) = r$, $(M, {ds_M}^2)$ is the Euclidean space $\mathbb{R}^2$.

From now on we will restrict our discussion to two dimensional manifolds. Let $(M, {ds_M}^2)$ and $(N, {ds_N}^2)$ be two radially symmetric Riemannian manifolds of dimension two with metrics in polar coordinate

$${ds_M}^2 = dr^2 + m(r)^2 d\theta^2, \quad {ds_N}^2 = dR^2 + n(R)^2 d\phi^2.$$

A map $f : M \to N$ is said to be *radially symmetric* if it is equivariant under the rotations with respect to poles or equivalently $R \circ f(r, \theta)$ depends only on r and $\phi \circ f(r, \theta) = \theta$.

We set, for $0 < a < b$, the annulus in M as $A(a, b) = \{(r, \theta) \in M : a \le r \le b\}$ and the geodesic R_0-ball in N as $B(R_0) = \{(R, \phi) \in N : 0 \le R \le R_0\}$.

For $f \in H^1(A(a, b), B(R_0))$, the energy $E(f)$ of f is given by

$$E(f) = \frac{1}{2} \int_A |\nabla f|^2.$$

Received August 20, 1993.

This work was supported in part by the Basic Science Research Institute Program, Ministry of Education, 1992, Project Number BSRI-92-107, by Seo Am Scholarship Foundation and by GARC.

We also set

$$\mathcal{F}(A(a,b),B(R_0)) \\ =\{f \in H^1(A(a,b),B(R_0)) : f(r,\theta) = (R_0,\theta) \text{ for all } (r,\theta) \in \partial A(a,b)\}.$$

F. Bethuel, H. Brezis, B. D. Colman and F. Hélein studied in [1] the cases when $A(\rho,1) \subset \mathbb{R}^2$ is an annulus in the Euclidean plane and $B(\pi/2) \subset S^2$ is the hemi-sphere with standard metric, and proved that there is a unique energy minimizer in the class of radially symmetric maps in $\mathcal{F}(A(\rho,1),B(\pi/2))$ and if $\rho > e^{-\pi}$,

$$f(r,\theta) = (R_0,\theta) \text{ for each } (r,\theta) \in A(\rho,1)$$

is the only minimizer. É. Sandier [2] pointed out that in fact minimizers are radially symmetric.

We show that this bifurcation phenomena occurs in general for radially symmetric minimizing maps between radially symmetric manifolds.

THEOREM 1. *Assume that the curvature $k(R)$ of $(N, {ds_N}^2)$ is positive and $n(R)$ satisfies $n'(R_0) = 0$ and $n'''(R) \leq 0$ for $0 \leq R \leq R_0$. If $\frac{\pi}{\sqrt{C}} > \int_a^b \frac{1}{m(r)}dr$ for $C = n(R_0)^2 \left(k(R_0) - \left(\frac{n'(R_0)}{n(R_0)}\right)^2\right)$, then the only minimizer f in the class of radially symmetric maps in $\mathcal{F}(A(a,b),B(R_0))$ is $f(r,\theta) = (R_0,\theta)$.*

Proof. An energy minimizer $f \in \mathcal{F}(A(a,b),B(R_0))$ is smooth and its energy is given by

$$\begin{aligned} E(f) &= \frac{1}{2}\int_A |\nabla f|^2 \\ &= \frac{1}{2}\int_0^{2\pi}\int_a^b \left\{ \left(\left|\frac{\partial(R\circ f)}{\partial r}\right|^2 + \frac{1}{m(r)^2}\left|\frac{\partial(R\circ f)}{\partial\theta}\right|^2 \right) \right. \\ &\qquad \left. + n(R\circ f)^2 \left(\left|\frac{\partial(\phi\circ f)}{\partial r}\right|^2 + \frac{1}{m(r)^2}\left|\frac{\partial(\phi\circ f)}{\partial\theta}\right|^2 \right) \right\} m(r)drd\theta. \end{aligned}$$

We change variables by

$$s(r) = \int_a^r \frac{d\rho}{m(\rho)},$$

to get

$$E(f)=\frac{1}{2}\int_0^{2\pi}\int_0^{s(b)}\left\{\left(\left|\frac{\partial(R\circ f)}{\partial s}\right|^2+\left|\frac{\partial(R\circ f)}{\partial\theta}\right|^2\right)\right.$$
$$\left.+n(R\circ f)^2\left(\left|\frac{\partial(\phi\circ f)}{\partial s}\right|^2+\left|\frac{\partial(\phi\circ f)}{\partial\theta}\right|^2\right)\right\}dsd\theta.$$

For a radially symmetric f, we have

$$E(f)=\pi\int_0^{s(b)}\left\{\left|\frac{\partial(R\circ f)}{\partial s}\right|^2+n(R\circ f)^2\right\}ds.$$

Putting $y=R_0-R\circ f$, we write the Euler-Lagrange equation for minimizing y as

$$\begin{aligned}&y''+g(y)=0\quad\text{for } g(y)=n(R_0-y)n'(R_0-y),\\&y(0)=y(s(b))=0,\\&0\le y\le R_0.\end{aligned}$$

Assume that $n'''(R)\le 0$ for $0\le R\le R_0$.

By the above assumption we have that $g''(y)<0$ for $0\le y\le R_0$, and so $g(y)\le Cy$ where $C=g'(0)=n(R_0)^2\left(k(R_0)-\left(\frac{n'(R_0)}{n(R_0)}\right)^2\right)$.

We use the following Lemma 2 to show that if y is not identically 0, $y(r)\ge\frac{y'(0)}{\sqrt{C}}\sin(\sqrt{C}r)$ for $r<\frac{\pi}{\sqrt{C}}$. Thus $y(s(b))\ge\frac{y'(0)}{\sqrt{C}}\sin(\sqrt{C}r)>0$, which contradicts to the boundary condition. We have that y is identically 0 and complete the proof of Theorem 1.

LEMMA 2. *Let f, $g:[0,R_0]\to[0,\infty)$ be continuous functions with $f\ge g$. Then the solutions y_1, y_2 of the initial value problem*

$$\begin{aligned}&y_1''(r)+f(y_1(r))=0,\\&y_2''(r)+g(y_2(r))=0,\\&y_1(0)=y_2(0)=0,\quad y_1'(0)=y_2'(0),\end{aligned}$$

satisfies $y_1(r) \leq y_2(r)$ for $r \leq r_0$, where $r_0 > 0$ is such that $y_1(r) > 0$ for $r < r_0$.

Proof. Let $r_0 = \min\{ r \mid r = R_0 \text{ or } y_1(r) = 0 \}$ and

$$L = \{ r \leq r_0 \mid y_1(r') \leq y_2(r'), \text{ for all } 0 \leq r' \leq r\}.$$

Put $x = \sup L$. If $x < r_0$, we have $y_1(x) = y_2(x)$. Multiplying the formula by y_1' and integrating it, we get

$$\begin{aligned}\frac{1}{2}(y_1')^2(x) - \frac{1}{2}(y_1')^2(0) &= -\int_0^{y_1(x)} f(y)dy \\ &\leq -\int_0^{y_1(x)} g(y)dy \\ &= \frac{1}{2}(y_2')^2(x) - \frac{1}{2}(y_2')^2(0).\end{aligned}$$

Thus $y_1'(x) \leq y_2'(x)$. Since $y_1(x) = y_2(x)$ and $y_1''(x) \leq y_2''(x)$, there is $\varepsilon > 0$ such that $y_1(r) \leq y_2(r)$ for $x \leq r \leq x + \varepsilon < r_0$. This contradicts to the definition of x. Thus we have $x = r_0$.

By using Theorem 1, we can prove the following result of F. Bethuel, H. Brezis, B. D. Colman and F. Hélein [1].

COROLLARY 3 [1]. *If $A(\rho, 1) \subset \mathbb{R}^2$ is an annulus in the Euclidean plane and $B(\pi/2) \subset S^2$ is the hemi-sphere with standard metric, and if $\rho > e^{-\pi}$, $f(r,\theta) = (R_0, \theta)$, for all $(r, \theta) \in A(\rho, 1)$, is the only minimizer in the class of radially symmetric maps in $\mathcal{F}(A(\rho,1), B(\pi/2))$.*

Proof. Let the metric of the annulus and the 2-sphere be

$$\begin{aligned} ds_M{}^2 &= dr^2 + r^2 d\theta^2, \\ ds_N{}^2 &= dR^2 + \sin^2 R d\phi^2, \end{aligned}$$

respectively. For $R_0 = \frac{\pi}{2}$, corresponding to the equator of S^2, we have $C = 1$. Thus if $\pi > \int_\rho^1 \frac{1}{r} dr$, or $\rho > e^{-\pi}$, then $f(r,\theta) = (\frac{\pi}{2}, \theta)$ is the only radially symmetric minimizer in $\mathcal{F}(A(\rho,1), B(\pi/2))$.

Remark. The proof of Theorem 1 shows that we may assume that the metric of the domain of f is equal to the Euclidean one and the bifurcation phenomena depends mainly on the property of the target manifold. Furthermore we can prove that an energy minimizer is radially symmetric if the target manifold is 2-sphere by using the argument of Sandier [2].

References

1. F. Bethuel, H. Brezis, B. D. Coleman, and F. Hélein, *Bifurcation analysis of minimizing harmonic maps describing the equilibrium of nematic phases between cylinders*, Arch. Rational Mech. Anal. **118** (1992), 149–168.
2. E. Sandier, *Symétrie des applications harmoniques minimisantes d'une couronne vers la sphère*, C. R. Acad. Sci. Paris, Sér I Math. **313** (1991), 435–440.

Department of Mathematics
Seoul National University
Seoul 151-742, Korea

Department of Mathematics
Hanshin University
Osan 447-791, Korea