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EXISTENCE OF RESONANCES FOR DIFFERENTIAL OPERATORS

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1. Introduction

Let H be a Schrödinger operator in $L^2(\mathbb{R})$

$$H = -\frac{d^2}{dx^2} + V(x),$$

with $\operatorname{supp} V \subset [-R, R]$. A number z_0 in the lower half-plane is called a resonance for H if for all φ with compact support $\langle \varphi, (H - z)^{-1}\varphi \rangle$ has an analytic continuation from the upper half-plane to a part of the lower half-plane with a pole at $z = z_0$. Thus a resonance is a sort of generalization of an eigenvalue. For $\operatorname{Im} k > 0$, $(H - k^2)^{-1}$ is an integral operator with kernel, given by Green's function

$$g(k,x,y) = \begin{cases} -\dfrac{\psi_+(k,x)\psi_-(k,y)}{W(k)}, & x \geq y \\ -\dfrac{\psi_-(k,x)\psi_+(k,y)}{W(k)}, & x \leq y, \end{cases}$$

where

$$-\psi''_\pm(k,x) + V(x)\psi_\pm = k^2\psi_\pm(k,x),$$

$$\psi_\pm(x) = e^{\pm ikx}, \qquad \pm x \geq R$$

and $W(k) = \psi'_+(k,x)\psi_-(k,x) - \psi_+(k,x)\psi'_-(k,x)$, which is independent of x.

Thus

$$\langle \varphi, (H - k^2)^{-1}\varphi \rangle = \iint g(k,x,y)\bar{\varphi}(x)\varphi(y)\,dx\,dy$$

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has an analytic continuation to the whole lower half-plane with poles where $W(k) = 0$, i.e., for k such that $\psi_+(k,x) = c\psi_-(k,x)$. Therefore k^2 is a resonance for H if and only if there exists ψ such that

$$-\psi'' + V\psi = k^2\psi, \qquad -\infty < x < \infty,$$

and $\psi(x) = C_\pm e^{\pm ikx}$ for $\pm x \geq R$. (outgoing conditions)

The simplest situation producing resonances near $E > 0$ is when $V(x)$ has "barriers" that trap classical particles of energy E, i.e., an interval $[a,b]$ where $V(x) < E$ is surrounded by classically forbidden regions (barriers) where $V(x) > E$. In quantum mechanics it is known that solutions eventually escape from such a trap.

2. Preliminaries

Suppose that V is a positive real-valued function supported in a finite interval $[-R,R]$ and that $E_0 = k_0^2\,(k_0 > 0)$ is the lowest eigenvalue of

$$H_N = -\frac{d^2}{dx^2} + V(x) = -\Delta + V(x)$$

on $L^2(-R,R)$ with Neumann boundary conditions at $x = \pm R$. So there is a solution φ of the eigenvalue equation $H_N\varphi = k_0^2\varphi$ with $\varphi'(\pm R) = 0$. We will sketch the proof of existence of resonance for H at k near k_0 where $k_0^2 = E_0$ is the lowest eigenvalue of Neumann operator H_N if the barrier is large enough, i.e., if $V(x) - E_0$ is large enough in the classically forbidden regions, and estimate $|k - k_0|$.

LEMMA 2.1. *Suppose $E(\beta)$ satisfies*

$$-\psi_\beta'' + V\psi_\beta = E(\beta)\psi_\beta$$

with $\psi_\beta'(\pm R) = \pm\beta\psi_\beta(\pm R)$, where $E(\beta)$ and ψ_β vary continuously with β. Then

$$\frac{dE(\beta)}{d\beta} = -\frac{\psi_\beta(R)^2 + \psi_\beta(-R)^2}{\int_{-R}^{R} \psi_\beta(x)^2\,dx}. \tag{2.1}$$

REMARK. The idea for the proof of existence of resonance is that if $|\psi_\beta(\pm R)|^2$ is small compared to $\int_{-R}^{R} \psi_\beta(x)^2\,dx$, then, by this lemma, E varies slowly as β changes. Since a resonance k is equivalent to a root of $E(ik) = k^2$, we will show that by Rouché's theorem, there is a k_* inside a circle centered at $k_0 = \sqrt{E_0}$ for which $E(ik_*) = k_*^2$ as long as $V(x) - E_0$ is sufficiently large in the forbidden region.

Proof. Suppose that for $i = 1, 2$

$$-\psi''_{\beta_i} + V\psi_{\beta_i} = E(\beta_i)\psi_{\beta_i}$$

for $|x| < R$ with $\psi'_{\beta_i}(\pm R) = \pm\beta_i\psi_{\beta_i}(\pm R)$. Then

$$\begin{aligned}
[E(\beta_1) - E(\beta_2)] \int_{-R}^{R} \psi_{\beta_1}\psi_{\beta_2}\,dx &= \int_{-R}^{R} (\psi''_{\beta_2}\psi_{\beta_1} - \psi''_{\beta_1}\psi_{\beta_2})\,dx \\
&= \int_{-R}^{R} \frac{d}{dx}(\psi'_{\beta_2}\psi_{\beta_1} - \psi'_{\beta_1}\psi_{\beta_2})\,dx \\
&= (\psi'_{\beta_2}\psi_{\beta_1} - \psi'_{\beta_1}\psi_{\beta_2})\big|_{-R}^{R} \\
&= (\beta_2 - \beta_1)[\psi_{\beta_2}(R)\psi_{\beta_1}(R) \\
&\qquad + \psi_{\beta_2}(-R)\psi_{\beta_1}(-R)].
\end{aligned}$$

Divide by $\beta_1 - \beta_2$ and take limit as $\beta_1 - \beta_2 \to 0$. Then we have

$$\frac{dE}{d\beta} = -\frac{\psi_\beta(R)^2 + \psi_\beta(-R)^2}{\int_{-R}^{R} \psi_\beta^2\,dx}.$$

3. Estimates for eigenfunctions in the forbidden region

Now we need to prove estimates to show (2.1) is small. To show that $\psi(\pm R)$ is small we use the fact in [1], [4] that eigenfunctions are small in the "forbidden region", $\{x : V(x) > \operatorname{Re} E\}$. Suppose $\{x : V(x) \le \operatorname{Re} E\}$ is an interval in $[-R, R]$.

LEMMA 3.1. *If* $-\psi'' + V\psi = E\psi$ *on* $[-R, R]$ *and* $F(x) \equiv 1$ *when* $V(x) < \operatorname{Re} E$,

$$\int_{V>\operatorname{Re} E} [(V - \operatorname{Re} E)|\psi|^2 + |\psi'|^2](F - \frac{|F'|}{2\sqrt{V - \operatorname{Re} E}})\, dx \leq \int_{V<\operatorname{Re} E} (\operatorname{Re} E - V)|\psi|^2\, dx + \left(F(x)\operatorname{Re}\psi'\bar{\psi}\right)\Big|_{-R}^{R}. \tag{3.1}$$

Proof. We have

$$\begin{aligned}
(F(x)\operatorname{Re}\psi'\bar{\psi})\Big|_{-R}^{R} &= \int_{-R}^{R} \frac{d}{dx}\Big(F\frac{\psi'\bar{\psi} + \bar{\psi}'\psi}{2}\Big)\, dx \\
&= \int_{-R}^{R} \Big(F'\frac{\psi'\bar{\psi} + \bar{\psi}'\psi}{2} + F\frac{\psi''\bar{\psi} + \bar{\psi}''\psi + 2|\psi'|^2}{2}\Big)\, dx \\
&= \int_{-R}^{R} \Big\{F'\frac{\psi'\bar{\psi} + \bar{\psi}'\psi}{2} + F[(V - \operatorname{Re} E)|\psi|^2 + |\psi'|^2]\Big\}\, dx \\
&\geq \int_{-R}^{R} [(V - \operatorname{Re} E)|\psi|^2 + |\psi'|^2](F - \frac{|F'|}{2\sqrt{V - \operatorname{Re} E}})\, dx
\end{aligned}$$

by using inequality $ab = (a\sqrt{c})(b/\sqrt{c}) \geq -\frac{a^2c^2+b^2}{2c}$, taking $c = \sqrt{V - \operatorname{Re} E}$. This implies

$$\int_{V>\operatorname{Re} E} [(V - \operatorname{Re} E)|\psi|^2 + |\psi'|^2](F - \frac{|F'|}{2\sqrt{V - \operatorname{Re} E}})\, dx \leq \left(F(x)\operatorname{Re}\psi'\bar{\psi}\right)\Big|_{-R}^{R} + \int_{V<\operatorname{Re} E} (\operatorname{Re} E - V)|\psi|^2\, dx.$$

Suppose $\{x : V(x) \leq \operatorname{Re} E\} \subset [R_-, R_+] \subset (-R, R)$. Choose $\delta_+ > 0$, $\delta_- > 0$, so

$$\int_{R_+}^{R-\delta_+} \sqrt{V(s) - \operatorname{Re} E}\, ds = \int_{-R+\delta_-}^{R_-} \sqrt{V(s) - \operatorname{Re} E}\, ds \equiv b$$

Define F as follows;

$$F(x)=\begin{cases} 1 & \text{on} \quad R_- \le x \le R_+ \\ \exp(2\int_{R_+}^{x} \sqrt{V(s)-\operatorname{Re}E}\,ds) & \text{on} \quad R_+ < x \le R-\delta_+ \\ \exp(-2\int_{R_-}^{x} \sqrt{V(s)-\operatorname{Re}E}\,ds) & \text{on} \quad -R+\delta_- \le x < R_- \\ \exp(2b) \quad \text{on} \quad R-\delta_+ < x \quad \text{or} \quad x < -R+\delta_-. \end{cases}$$

Then, clearly

$$\frac{|F'(x)|}{2\sqrt{V-\operatorname{Re}E}} = F \quad \text{for} \quad R_+ < x \le R-\delta_+ \quad \text{or} \quad -R+\delta_- \le x < -R.$$

Thus, if

$$B(E)=\exp(2b)=\exp\bigl(\pm 2\int_{R_\pm}^{\pm R\mp\delta_\pm} \sqrt{V(s)-\operatorname{Re}E}\,ds\bigr),$$

Lemma 3.1 implies

$$\text{(3.2)} \quad \Bigl(\int_{R-\delta_+}^{R} + \int_{-R}^{-R+\delta_-}\Bigr)[(V-\operatorname{Re}E)|\psi|^2 + |\psi'|^2]\,dx \\ \le (\operatorname{Re}\psi'\bar{\psi})\big|_{-R}^{R} + B(E)^{-1}\int_{R_-}^{R_+} (\operatorname{Re}E - V)|\psi|^2\,dx.$$

This tells us that the integrals of $|\psi|^2$ and $|\psi'|^2$ are small near $x=\pm R$ if the integral of $\sqrt{V(s)-\operatorname{Re}E}$ is large over the forbidden region, so its exponential is very large and if $\operatorname{Re}\psi'\bar{\psi}$ is small at $x=\pm R$.

We need to know that $|\psi(\pm R)|^2$ is small if the integral in the right hand side of (3.2) is small. For this we may use the following inequality.

LEMMA 3.2. *With V, $\delta_\pm$, and ψ as above*

$$\text{(3.3)} \quad |\psi(R)|^2 + |\psi(-R)|^2 \le \frac{1}{a}\frac{e^2+1}{e^2-1}(\operatorname{Re}\psi'\bar{\psi})\big|_{-R}^{R} \\ + \frac{B(E)^{-1}}{a}\frac{e^2+1}{e^2-1}\sup(\operatorname{Re}E - V)\int_{R_-}^{R_+} |\psi|^2\,dx$$

if $a^2 \le V - \operatorname{Re} E$ *for* $R - \delta_+ \le x \le R$, *and* $-R \le x \le -R + \delta_-$ *with* $\delta_\pm \ge \frac{1}{a}$.

Proof. We have, for real f

$$f(x)|\varphi(x)|^2\Big|_\alpha^\beta = \int_\alpha^\beta \frac{d}{dx}(f|\varphi|^2)\,dx = \int_\alpha^\beta [f'|\varphi|^2 + f(\varphi'\bar\varphi + \varphi\bar\varphi')]\,dx \le \int_\alpha^\beta [(f' + f^2)|\varphi|^2 + |\varphi'|^2]\,dx. \tag{3.4}$$

If we choose $f(x) = a\tanh\bigl[a(x - (R - \delta_+))\bigr]$ and $\alpha = R - \delta_+$, $\beta = R$, then $f(\alpha) = 0$, $f(\beta) = a\tanh(a\delta_+) > a\tanh 1$, and $f' + f^2 = a^2(\operatorname{sech}^2 + \tanh^2)\bigl[a(x - (R - \delta_+))\bigr] = a^2$. So if $a^2 \le V - \operatorname{Re} E$ for $R - \delta_+ \le x \le R$, we obtain

$$\begin{aligned} f(x)\ |\psi(x)|^2\Big|_\alpha^\beta &= a\tanh(a\delta_+)|\psi(R)|^2 \\ &\le \int_{R-\delta_+}^R [a^2|\psi|^2 + |\psi'|^2]\,dx \qquad \text{by (3.4)} \\ &\le \int_{R-\delta_+}^R [(V - \operatorname{Re} E)|\psi|^2 + |\psi'|^2]\,dx. \end{aligned}$$

Similarly, if $a^2 \le V - \operatorname{Re} E$ for $-R \le x \le -R + \delta_-$,

$$\begin{aligned} f(x)\ |\psi(x)|^2\Big|_\alpha^\beta &= a\tanh(a\delta_-)|\psi(-R)|^2 \\ &\le \int_{-R}^{-R+\delta_-} [a^2|\psi|^2 + |\psi'|^2]\,dx \\ &\le \int_{-R}^{-R+\delta_-} [(V - \operatorname{Re} E)|\psi|^2 + |\psi'|^2]\,dx. \end{aligned}$$

(by choosing $f = a\tanh\bigl[a(x - (-R + \delta_-))\bigr]$, and $\alpha = -R$, $\beta = -R + \delta_-$.)

Since $a\delta_\pm \ge 1$, we have

$$\tanh(a\delta_\pm) \ge \tanh 1 = \frac{e - e^{-1}}{e + e^{-1}} = \frac{e^2 - 1}{e^2 + 1}.$$

Thus (3.2) yields

$$|\psi(R)|^2 + |\psi(-R)|^2 \le \frac{1}{a}\frac{e^2+1}{e^2-1}(\operatorname{Re}\psi'\bar{\psi})\Big|_{-R}^{R} + \frac{B(E)^{-1}}{a}\frac{e^2+1}{e^2-1}\int_{R_-}^{R_+}(\operatorname{Re}E - V)|\psi|^2\,dx$$

if $a^2 \le V - \operatorname{Re}E$ for $R-\delta_+ \le x \le R$, and $-R \le x \le -R+\delta_-$ with $\delta_\pm \ge \frac{1}{a}$. This completes the proof of this lemma.

We need (3.3) for the values of E near E_0. Let us get a certain inequality that does not depend on E for E near E_0.

We know that

$$\sqrt{x} = \sqrt{x_0 + x - x_0} \ge \sqrt{x_0} - \frac{|x-x_0|}{\sqrt{x_0}}.$$

This implies, if $V > \operatorname{Re}E$

$$\begin{aligned}\sqrt{V-\operatorname{Re}E} &= \sqrt{V - E_0 + V - \operatorname{Re}E - (V-E_0)}\\ &\ge \sqrt{V-E_0} - \frac{|\operatorname{Re}E - E_0|}{\sqrt{V-E_0}}\end{aligned}$$

on $[R_+, R-\delta_+]$ so that

$$\int_{R_+}^{R-\delta_+}\sqrt{V(s)-\operatorname{Re}E}\,ds \ge \int_{R_+}^{R-\delta_+}\Big(\sqrt{V(s)-E_0} - \frac{|\operatorname{Re}E-E_0|}{\sqrt{V(s)-E_0}}\Big)\,ds.$$

Hence if $V(x) > \operatorname{Re}E$ for $x \in [R_+, R]$ and

$$\int_{R_+}^{R-\delta_+}\frac{|\operatorname{Re}E-E_0|}{\sqrt{V(s)-E_0}}\,ds \le \frac{1}{2}, \tag{3.5}$$

$$\int_{R_+}^{R-\delta_+}\sqrt{V(s)-\operatorname{Re}E}\,ds \ge \int_{R_+}^{R-\delta_+}\sqrt{V(s)-E_0}\,ds - \frac{1}{2}.$$

That is,

$$\exp(2\int_{R_+}^{R-\delta_+}\sqrt{V(s)-\operatorname{Re}E}\,ds)\geq e^{-1}\exp(2\int_{R_+}^{R-\delta_+}\sqrt{V(s)-E_0}\,ds).$$

Similarly, on $[-R+\delta_-, R_-]$

$$\begin{aligned}&\exp(-2\int_{R_-}^{-R+\delta_-}\sqrt{V(s)-\operatorname{Re}E}\,ds)\\ &\geq e^{-1}\exp(-2\int_{R_-}^{-R+\delta_-}\sqrt{V(s)-E_0}\,ds)\end{aligned}$$

if

$$\int_{-R+\delta_-}^{R_-}\frac{|\operatorname{Re}E-E_0|}{\sqrt{V(s)-E_0}}\,ds\leq\frac{1}{2}. \tag{3.5a}$$

Therefore, since $B(E)=\exp\left(\pm 2\int_{R_\pm}^{\pm R\mp\delta_\pm}\sqrt{V(s)-\operatorname{Re}E}\,ds\right)$, we have

$$B(E)\geq\frac{B(E_0)}{e},$$

where

$$B(E_0)=\min\left\{\exp(2\int_{R_+}^{R-\delta_+}\sqrt{V(s)-E_0}\,ds),\ \exp(-2\int_{R_-}^{-R+\delta_-}\sqrt{V(s)-E_0}\,ds)\right\}, \tag{3.6}$$

if (3.5) and (3.5a) are satisfied.

Let us assume

(3.7)

$$\begin{cases}|\operatorname{Re}E-E_0|\leq\frac{1}{2}\min\left\{\inf_{[R_+,R]}\frac{\sqrt{V(x)-E_0}}{R-R_+},\ \inf_{[-R,R_-]}\frac{\sqrt{V(x)-E_0}}{R_-+R}\right\},\\ \text{and}\\ \inf_{[R_+,R]}\sqrt{V-E_0}(R-R_+)>2,\qquad \inf_{[-R,R_-]}\sqrt{V-E_0}(R_-+R)>2.\end{cases}$$

Then (3.5) and (3.5a) above are both satisfied, and moreover,

$$
\begin{aligned}
|\operatorname{Re} E - E_0| &\le \frac{1}{2}\min\left\{ \inf_{[R_+,R]} \frac{V-E_0}{\sqrt{V-E_0}(R-R_+)}, \right. \\
&\qquad \left. \inf_{[-R,R_-]} \frac{V-E_0}{\sqrt{V-E_0}(R_-+R)} \right\} \\
&\le \inf \frac{V-E_0}{4} \text{ for } -R \le x \le R_- \text{ and} \\
&\qquad R_+ \le x \le R, \text{ so that } V - \operatorname{Re} E > 0
\end{aligned}
$$

and also

$$
\sqrt{V - \operatorname{Re} E} \ge \sqrt{V-E_0} - \frac{|\operatorname{Re} E - E_0|}{\sqrt{V-E_0}} \ge \frac{3}{4}\sqrt{V-E_0}
$$

for $R-\delta_+ \le x \le R$ and $-R \le x \le -R+\delta_-$. Combining the above with Lemma 3.2 gives the following.

PROPOSITION 3.3. *If* $-\psi'' + V\psi = E\psi$ *on* $[-R,R]$ *and* E_0 *is the lowest eigenvalue for Neumann operator* $H_N = -\frac{d^2}{dx^2} + V$ *on* $[-R,R]$,

$$
\begin{aligned}
(3.8)\quad |\psi(R)|^2 + |\psi(-R)|^2 &\le \frac{1}{a}\frac{e^2+1}{e^2-1}(\operatorname{Re}\psi'\bar{\psi})\Big|_{-R}^{R} \\
&+ \frac{e}{a}\frac{e^2+1}{e^2-1}B(E_0)^{-1}\sup(E_0-V)\Big(1+\frac{|\operatorname{Re} E - E_0|}{\sup(E_0-V)}\Big)\int_{-R}^{R}|\psi|^2\,dx,
\end{aligned}
$$

provided $a \le \frac{3}{4}\sqrt{V-E_0}$ *for* $R-\delta_+ \le x \le R$ *and* $-R \le x \le -R+\delta_-$ *with* $a\delta_\pm \ge 1$ *and* (3.7) *holds, with* $B(E_0)$ *given by* (3.6).

REMARK. The condition that $\sqrt{V(x)-E_0}$ is bounded below on $[R_+,R]$ and $[-R,R_-]$ required by (3.7) means that V must be discontinuous at $\pm R$ in order to have support $[-R,R]$. Dropping these assumptions would introduce extra complications.

Now, since the denominator in (2.1) is $\int_{-R}^{R}\psi^2\,dx$, not $\int_{-R}^{R}|\psi|^2\,dx$, we need a lower bound for $|\int_{-R}^{R}\psi^2\,dx|$. If $-\psi''+V\psi = E\psi$ with E real and real boundary conditions, then $\psi \pm \bar{\psi}$ solves the same problem, so we could choose a real eigenfunction. But in the general case we are dealing with, this is not so, and $\int_{-R}^{R} \psi^2\,dx$ might be small or even vanish.

Let $H_N\varphi_n = -\varphi_n'' + V\varphi_n = E_n\varphi_n$ with $\varphi_n'(\pm R) = 0,\ n = 0, 1, \ldots$. It can be shown that $E_0 < E_1 < \cdots$. We can argue that if k^2 is close to E_0, then a solution ψ of $H\psi = -\psi'' + V\psi = k^2\psi$ will be close to $C\varphi_0$. Let P be the spectral projection for H_N and the interval (E_0, ∞). Then

$$P = \chi_{(E_0,\infty)}(H_N) = \chi_{[E_1,\infty)}(H_N)$$

and since

$$\chi_{[E_1,\infty)}(\lambda) \le \frac{\lambda - E_0}{E_1 - E_0} \qquad \text{for} \qquad \lambda \ge E_0,$$

we have for $\varphi \in \mathfrak{D}(H_N)$,

$$\begin{aligned}\|P\varphi\|^2 &\le <\varphi, \frac{H_N - E_0}{E_1 - E_0}\varphi> \\ &= \frac{\int_{-R}^{R}(-\varphi''\bar{\varphi} + V|\varphi|^2 - E_0|\varphi|^2)\,dx}{E_1 - E_0} \\ &= \frac{\int_{-R}^{R}(|\varphi'|^2 + (V - E_0)|\varphi|^2)\,dx}{E_1 - E_0},\end{aligned}$$

since $\varphi'(\pm R) = 0$. Now for $\psi \in \mathfrak{D}\big((H_N - E_0 + 1)^{\frac{1}{2}}\big) = \mathcal{Q}$, since $\mathfrak{D}(H_N)$ is dense in $\mathcal{Q}$, the above formula holds as well.

$$\begin{aligned}\|P\psi\|^2 &\le \int_{-R}^{R}(|\psi'|^2 + (V - E_0)|\psi|^2)\,dx\,/(E_1 - E_0) \\ &= \frac{\int_{-R}^{R}(-\psi'' + V\psi - E_0\psi)\bar{\psi}\,dx + \psi'\bar{\psi}\big|_{-R}^{R}}{E_1 - E_0} \\ &= \frac{(k^2 - E_0)\|\psi\|^2 + \psi'\bar{\psi}\big|_{-R}^{R}}{E_1 - E_0}.\end{aligned}$$

Taking real parts gives

$$\|P\psi\|^2 \le \frac{(\operatorname{Re} k^2 - E_0)\|\psi\|^2 + \operatorname{Re}\psi'\bar{\psi}\big|_{-R}^{R}}{E_1 - E_0}. \tag{3.9}$$

Since we have $\psi = \langle \varphi_0, \psi \rangle \varphi_0 + P\psi$, choosing φ_0 real and choosing ψ such that $\langle \varphi_0, \psi \rangle$ is real gives

$$\begin{aligned}\int_{-R}^{R} \psi^2 \, dx = \langle \varphi_0, \psi \rangle^2 \int_{-R}^{R} |\varphi_0|^2 \, dx \\ + 2\langle \varphi_0, \psi \rangle \int_{-R}^{R} \varphi_0 P\psi \, dx + \int_{-R}^{R} (P\psi)^2 \, dx \\ \geq \langle \varphi_0, \psi \rangle^2 - \|P\psi\|^2 \qquad \text{since} \quad \langle \varphi_0, P\psi \rangle = 0.\end{aligned}$$

Hence

$$\begin{aligned}\left| \int_{-R}^{R} \psi^2 \, dx \right| &\geq \langle \varphi_0, \psi \rangle^2 - \|P\psi\|^2 \\ &= \|\psi\|^2 - 2\|P\psi\|^2.\end{aligned}$$

Thus from this inequality with (3.9) we obtain

PROPOSITION 3.4. *If $-\psi'' + V\psi = k^2\psi$ on $[-R, R]$ and E_0 and E_1 $(E_0 < E_1)$ are the two lowest eigenvalues of H_N,*

$$\left| \int_{-R}^{R} \psi^2 \, dx \right| \geq \|\psi\|^2 \Big(1 - \frac{2|\operatorname{Re} k^2 - E_0|}{E_1 - E_0}\Big) - \frac{2\operatorname{Re} \psi' \bar{\psi} \big|_{-R}^{R}}{E_1 - E_0}. \tag{3.10}$$

In fact a lower bound for $E_1 - E_0$ can be derived for H_N. Assume that the Schrödinger operator $H = -\frac{d^2}{dx^2} + V$ has a symmetric single-well potential V so that $V(-x) = V(x)$ and $xV'(x) \geq 0$ for $|x| \leq R$.

The next lemma will deal with a lower bound for $E_1 - E_0$ for H_D with Dirichlet boundary conditions at $x = \pm R$. It appears in [2].

LEMMA 3.5. *Let $H_D = -\frac{d^2}{dx^2} + V(x)$ on $L^2(-R, R)$ with Dirichlet boundary conditions at $x = \pm R$. Suppose V is a symmetric single-well potential so that $V(-x) = V(x)$ and $xV'(x) \geq 0$ for $|x| \leq R$. If E_0 is the lowest eigenvalue and E_1, the next eigenvalue above E_0 for H_D, then*

$$E_1 - E_0 \geq \frac{3\pi^2}{(2R)^2}, \tag{3.11}$$

with equality if and only if V is constant.

Note that eigenfunctions of Dirichlet Laplacian H_D and Neumann Laplacian H_N are close together if $V(\pm R) >> E_0$. Further notice that for $\widetilde{\varphi} \in \mathfrak{D}(H_N)$, $\widetilde{\varphi}'(\pm x)$ is small near boundary $x = \pm R$. For $E_1 - E_0$ with Neumann Laplacian H_N, we can change functions $\widetilde{\varphi}_i$ $(i = 0, 1)$ in the domain of H_N to functions φ_i in the domain of H_D near $x = \pm R$ with small error by setting $\chi(x) = 1$ for $-R+\delta_- \leq x \leq R-\delta_+$ and $\chi(\pm R) = 0$ and taking $\varphi_i = \chi\widetilde{\varphi}_i$. Hence H_N has eigenvalues E_i^N near E_i of H_D.

4. Existence of resonances for differential operators

In order to find k such that $E(ik) = k^2$, let us use Rouché's theorem: if f and g are analytic inside and on a circle γ, then f and g have the same number of zeros inside γ if $|f - g| < |f|$ on γ.

Take $f(k) = k^2 - E_0$ and $g(k) = k^2 - E(ik)$. We will show that $|f - g| < |f|$ on a circle with small radius centered at $k_0 = \sqrt{E_0}$. Then we will know that there is a k_* inside the circle for which $E(ik_*) = k_*^2$.

First we estimate $|E(ik_0) - E_0|$.

LEMMA 4.1. *If $-\psi'' + V\psi = E(ik_0)\psi$ on $[-R, R]$ with $\psi'(\pm R) = \pm ik_0\psi(\pm R)$, then*

$$|E(ik_0) - E_0| \leq 2k_0 \frac{e^2+1}{e^2-1}\frac{e}{a}B(E_0)^{-1}\sup(E_0 - V), \tag{4.1}$$

provided $2k_0 \frac{e^2+1}{e^2-1}\frac{e}{a}B(E_0)^{-1}(1 + 4\frac{\sup(E_0-V)}{E_1-E_0}) < 1$, where $k_0 = \sqrt{E_0} = \sqrt{E(0)}$, and $a \leq \frac{3}{4}\sqrt{V - E_0}$ for $R-\delta_+ \leq x \leq R$ and $-R \leq x \leq -R+\delta_-$ with $a\delta_\pm \geq 1$ and the right hand side of (4.1) is less than

$$C \equiv \frac{1}{2}\min\left\{\inf_{[R_+,R]} \frac{\sqrt{V(x) - E_0}}{R - R_+}, \inf_{[-R,R_-]} \frac{\sqrt{V(x) - E_0}}{R_- + R}\right\}.$$

Proof. We have, by Lemma 2.1,

$$\begin{aligned} \left|\frac{d}{dt}E(itk_0)\right| &= \left|k_0\frac{\psi(-R)^2 + \psi(R)^2}{\int_{-R}^{R}\psi^2\,dx}\right| \\ &\leq \frac{k_0 \frac{e^2+1}{e^2-1}\frac{e}{a}B(E_0)^{-1}\sup(E_0 - V)(1 + \frac{|\mathrm{Re}\,E(itk_0)-E_0|}{\sup(E_0-V)})\|\psi\|^2}{\|\psi\|^2(1 - 2\frac{|\mathrm{Re}\,E(itk_0)-E_0|}{E_1-E_0})} \end{aligned} \tag{4.2}$$

if (3.7) holds with $E = E(itk_0)$, using Proposition 3.3, Proposition 3.4 and the fact that $\operatorname{Re} \psi' \bar{\psi}|_{-R}^{R} = 0$, since $\psi'(\pm R) = \pm itk_0 \psi(\pm R)$.

In fact letting $y(t) = |E(itk_0) - E_0|$, we know by (4.2), $y(t)$ satisfies a differential inequality of the form

$$\frac{dy}{dt} \leq \frac{\epsilon + \alpha y}{1 - \beta y}, \qquad y(0) = 0,$$

as long as $y(t) \leq C$ so that (3.7) holds, where

$$\epsilon = k_0 \frac{e^2+1}{e^2-1} \frac{e}{a} B(E_0)^{-1} \sup(E_0 - V) \leq \frac{C}{2},$$
$$\alpha = \frac{\epsilon}{\sup(E_0 - V)}, \qquad \text{and} \qquad \beta = \frac{2}{E_1 - E_0}.$$

Here, ϵ will be small if the barrier, i.e., $V(x) - E_0$ is large enough in the forbidden region.
Assume that

$$2\alpha + 4\epsilon\beta \leq 1. \tag{4.3}$$

We claim that $y(1) \leq 2\epsilon$ if (4.3) holds: for t small, $y(t) \leq 2\epsilon$ and as long as $y(t) \leq 2\epsilon$, we have

$$y(t) \leq \frac{\epsilon + 2\epsilon\alpha}{1 - 2\epsilon\beta} t.$$

Thus, $y(t) \leq 2\epsilon$ for $0 \leq t \leq 1$ if $\frac{\epsilon+2\epsilon\alpha}{1-2\epsilon\beta} \leq 2\epsilon$, which is guaranteed by (4.3) above.
Therefore, the conclusion (4.1) follows,

$$|E(ik_0) - E_0| \leq 2k_0 \frac{e^2+1}{e^2-1} \frac{e}{a} B(E_0)^{-1} \sup(E_0 - V)$$

if $B(E_0)$ is large enough so that $2k_0 \frac{e^2+1}{e^2-1} \frac{e}{a} B(E_0)^{-1} \left(1 + \frac{4 \sup(E_0 - V)}{E_1 - E_0}\right) \leq 1$ so that (4.3) holds and the right hand side of (4.1) is less than C.

Now, we estimate $|f - g| = |E(ik) - E_0|$ by relating $E(ik)$ to $E(ik_0)$, and the existence of resonance is derived.

THEOREM 4.2. *Suppose V is a positive real-valued function with compact support in $[-R, R]$ and the operator $H_N = -\frac{d^2}{dx^2} + V(x)$ on $[-R, R]$ with Neumann boundary conditions has eigenvalues $k_0^2 = E_0 < E_1 < \cdots$. The operator $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R})$ has a resonance $E = k^2$ such that*

$$|k - k_0| < \frac{16}{7}\frac{e^2+1}{e^2-1}\frac{e}{a}B(E_0)^{-1}\sup(E_0 - V), \tag{4.4}$$

provided that $B(E_0)$ given in (3.6) is large enough so that the right hand side of (4.4) is less than $\frac{k_0}{4}$ and

$$4k_0\frac{e^2+1}{e^2-1}\frac{e}{a}B(E_0)^{-1}\sup(E_0 - V) < \frac{1}{10}(E_1 - E_0),$$

$$2k_0\frac{e^2+1}{e^2-1}\frac{e}{a}B(E_0)^{-1}\sup(E_0 - V) < \frac{1}{2}\min\left\{\inf_{[R_+,R]}\frac{\sqrt{V-E_0}}{R-R_+}, \inf_{[-R,R_-]}\frac{\sqrt{V-E_0}}{R_-+R}\right\},$$

$$2k_0\frac{e^2+1}{e^2-1}\frac{e}{a}B(E_0)^{-1}\left(1 + \frac{4\sup(E_0-V)}{E_1-E_0}\right) < 1.$$

Proof. Let $|\gamma| = 1$. Then by Lemma 2.1,

$$\begin{aligned}|E(i(k_0+\tau\gamma)) - E(ik_0)| &= \left|\int_0^\tau \frac{d}{dt}E(i(k_0+t\gamma))\,dt\right| \\ &= \left|\gamma\int_0^\tau \frac{\psi(R)^2+\psi(-R)^2}{\int_{-R}^R \psi^2\,dx}\,dt\right| \\ &\le \int_0^\tau \frac{|\psi(R)|^2+|\psi(-R)|^2}{|\int_{-R}^R \psi^2\,dx|}\,dt,\end{aligned}$$

with $\psi'(\pm R) = \pm ik\psi(\pm R)$, where $k = k_0 + t\gamma = \kappa - i\eta\,(\kappa, \eta > 0)$.

Furthermore, we have

$$
\begin{aligned}
\kappa(|\psi(R)|^2+|\psi(-R)|^2) &= \operatorname{Im}\,\psi'\bar{\psi}\Big|_{-R}^{R} \\
&= \int_{-R}^{R}\frac{d}{dx}\frac{\psi'\bar{\psi}-\psi\bar{\psi}'}{2i}\,dx \\
&= \int_{-R}^{R}\frac{\psi''\bar{\psi}-\psi\bar{\psi}''}{2i}\,dx \\
&= -\operatorname{Im}E\int_{-R}^{R}|\psi|^2\,dx = -\operatorname{Im}E\,\|\psi\|^2.
\end{aligned}
\tag{4.5}
$$

On the other hand,

$$
\operatorname{Re}\psi'\bar{\psi}\Big|_{-R}^{R} = \eta(|\psi(R)|^2+|\psi(-R)|^2) = -\frac{\eta}{\kappa}\operatorname{Im}E\,\|\psi\|^2
$$

by (4.5). Therefore, combining these with Proposition 3.4 gives

$$
\begin{aligned}
\frac{|\psi(R)|^2+|\psi(-R)|^2}{\left|\int_{-R}^{R}\psi^2\,dx\right|} &\le \frac{|\operatorname{Im}E|\cdot\|\psi\|^2}{\kappa\|\psi\|^2\left(1-2\frac{|\operatorname{Re}E-E_0|}{E_1-E_0}+2\frac{\eta}{\kappa}\frac{\operatorname{Im}E}{E_1-E_0}\right)} \\
&\le \frac{|\operatorname{Im}E|}{\kappa\left(1-2\frac{|E-E_0|}{E_1-E_0}(1+\frac{\eta^2}{\kappa^2})^{\frac12}\right)} \\
&= \frac{|\operatorname{Im}E|}{\kappa-2\frac{|E-E_0|}{E_1-E_0}|k|},
\end{aligned}
$$

since $|\operatorname{Re}E-E_0|-\frac{\eta}{\kappa}\operatorname{Im}E = \operatorname{Re}\left[(E-E_0)(1+i\frac{\eta}{\kappa})\right] \le |E-E_0|\sqrt{1+\frac{\eta^2}{\kappa^2}} = |E-E_0|\frac{|k|}{\kappa}$. Writing $k=k_0+t\gamma$ gives

$$
\frac{|\psi(R)|^2+|\psi(-R)|^2}{\left|\int_{-R}^{R}\psi^2\,dx\right|} \le \frac{|E-E_0|}{k_0-t-2\frac{|E-E_0|}{E_1-E_0}(k_0+t)}. \tag{4.6}
$$

As long as $t<\frac{k_0}{4}$ and $|E-E_0|<\frac{1}{10}(E_1-E_0)$, the denominator in (4.6) is

$$
k_0-t-2\frac{|E-E_0|}{E_1-E_0}(k_0+t) \ge \frac{k_0}{2}. \tag{4.6a}
$$

Hence, defining $\widetilde{E}(t) = E(i(k_0 + t\gamma))$ gives, by (4.6) and (4.6a),

$$|\widetilde{E}(\tau) - E(ik_0)| \leq \int_0^\tau \frac{|\psi(R)|^2 + |\psi(-R)|^2}{\left|\int_{-R}^R \psi^2\, dx\right|}\, dt$$

$$\leq 2\,\tau \sup_{t \leq \tau} \frac{|\widetilde{E}(t) - E_0|}{k_0}, \qquad \text{if} \qquad \tau < \frac{k_0}{4}. \tag{4.7}$$

Now, to estimate $|E(ik) - E_0|$ observe that, with

$$\epsilon = k_0 \frac{e^2+1}{e^2-1}\frac{e}{a} B(E_0)^{-1} \sup(E_0 - V),$$

$$\sup_{t\leq\tau} |\widetilde{E}(t) - E_0| \leq \sup_{t\leq\tau} |\widetilde{E}(t) - E(ik_0)| + |E(ik_0) - E_0|$$

$$\leq 2\,\tau \sup_{t\leq\tau} \frac{|\widetilde{E}(t) - E_0|}{k_0} + 2\epsilon,$$

by (4.7) and Lemma 4.1, which implies

$$|\widetilde{E}(\tau) - E_0|\left(1 - \frac{2\tau}{k_0}\right) < 2\epsilon,$$

i.e.,

$$|\widetilde{E}(\tau) - E_0| < 4\epsilon,$$

if $\tau < \frac{k_0}{4}$, $4\epsilon < \frac{1}{10}(E_1 - E_0)$, $2k_0 \frac{e^2+1}{e^2-1}\frac{e}{a} B(E_0)^{-1}\left(1 + \frac{4\sup(E_0 - V)}{E_1 - E_0}\right) < 1$ and $2\epsilon < C$ given in Lemma 4.1.

Finally we must show

$$|f - g| = |E(ik) - E_0| < |f| = |k - k_0|\,|k + k_0|$$

for $k = k_0 + \tau\gamma$ on a circle of radius τ.
On this circle,

$$|f| \geq \tau(2k_0 - \tau) \geq \frac{7}{4} k_0 \tau \qquad \text{if} \qquad \tau < \frac{k_0}{4}.$$

Thus $|f - g| < |f|$ as long as $4\epsilon < \frac{7}{4} k_0 \tau$, i.e.,

$$\tau > \frac{16}{7}\frac{e^2+1}{e^2-1}\frac{e}{a} B(E_0)^{-1} \sup(E_0 - V) \quad \text{if} \quad \tau < \frac{k_0}{4},$$

and the theorem is proved.

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