Comm. Korean Math. Soc. **9** (1994), No. 2, pp. 327–329

DIRICHLET PROBLEM ON THE UPPER HALF PLANE – A HEURISTIC ARGUMENT

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The Dirichlet problem (DP) on the upper half plane $\{z = x+iy : y > 0\}$ is to find a real-valued harmonic function $u(x,y)$ satisfying $u(x,0) = g(x)$ almost everywhere for some reasonably nice function g defined on the real line, which is called the data on the boundary for (DP). To find such a function we use the formula

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\xi)\,y}{(x-\xi)^2+y^2}d\xi \quad \text{for } y > 0.$$

In most references it is derived using Cauchy's integral formula. In this short article we derive the formula using elementary ideas. First we need

LEMMA. *Let g be a real valued function defined on the real line such that $g(x) = 1$ for $a \le x < b$ and $g(x) = 0$ elsewhere, i.e., $g(x) = \chi_{[a,b)}$. Choose a branch for $\log z$ so that it is single-valued and analytic on the upper half plane. For example, put $\log z = \log|z| + \mathrm{Arg}(z)$, $\frac{\pi}{2} <$ $Arg(z) < \frac{3\pi}{2}$. Then the solution of (DP) is given by*

$$u_{ab}(x,y) = \frac{1}{\pi}\,\mathrm{Im}\left[\log\frac{z-b}{z-a}\right].$$

Proof. Since $\mathrm{Arg}(z)$, $z \neq 0$ is the imaginary part of the analytic function $\log z$, it is harmonic. Hence $\mathrm{Arg}(z-b) - \mathrm{Arg}(z-a)$ is harmonic on the upper half plane, which is equal to the imaginary part of $\log(z-b) - \log(z-a) = \log\dfrac{z-b}{z-a}$. It is easy to see that the function satisfies the boundary condition except at $z = a, b$. For the details, see [1, p. 377].

Note that (DP) is linear with respect to the data g on the boundary in the sense that if u_1, u_2 are solutions for (DP) with boundary data

Received October 12, 1993.

g_1, g_2, respectively, then $u = c_1u_1 + c_2u_2$ is the solution for (DP) with boundary data $g = c_1g_1 + c_2g_2$ where c_1, c_2 are arbitrary constants.

For a general Dirichlet Problem we consider the case when g is piecewise continuous and integrable along the real line. We will generalize the concept of linearity of (DP) up to an infinite sum of g_i's and decompose the given data g into an infinite linear combination of characteristic functions of infinitesimally short intervals.

We partition the real axis into very short intervals $I_k = [x_k, x_{k+1})$, $-\infty < k < \infty$, and consider the (DP) for $g_k(x) \equiv g(x_k) \cdot \chi_{I_k}(x)$ and find the corresponding solution

$$u_k(z) \equiv g(x_k) \cdot \frac{1}{\pi} \operatorname{Im} \left[\log \frac{z - x_{k+1}}{z - x_k} \right].$$

Note that g is approximately the sum of all g_k since $\Delta x_k \equiv x_{k+1} - x_k$ is very small and g is continuous.

Since $\log \dfrac{z - x_{k+1}}{z - x_k} = \log(1 - \dfrac{\Delta x_k}{z - x_k})$ is approximately equal to $\dfrac{\Delta x_k}{x_k - z}$ by the first order approximation, $u_k(z)$ is approximately equal to $g(x_k) \cdot \dfrac{1}{\pi} \operatorname{Im} \left[\dfrac{\Delta x_k}{x_k - z} \right]$, hence the solution $u(z)$ of the original (DP) with the boundary data $g(x)$ is approximately equal to

$$\sum_{k=-\infty}^{\infty} g(x_k) \cdot \frac{1}{\pi} \operatorname{Im} \left[\frac{1}{x_k - z} \right] \Delta x_k.$$

As the partition of the real line becomes finer and finer, i.e., the lengths Δ_k get shorter indefinitely, we obtain

$$u(z) = \int_{-\infty}^{\infty} g(\xi) \cdot \frac{1}{\pi} \operatorname{Im} \left[\frac{1}{\xi - z} \right] d\xi.$$

Now we use $\operatorname{Im} \left[\dfrac{1}{\xi - z} \right] = \operatorname{Im} \left[\dfrac{1}{\xi - x - iy} \right] = \dfrac{y}{(\xi - x)^2 + y^2}$, which completes the proof.

REFERENCES

1. J. E. Marsden and M. J. Hoffman, *Basic Complex Analysis, 2nd ed.*, W. H. Freeman and Co., San Francisco, 1987.

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