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ON KATO'S DECOMPOSITION THEOREM

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1. Introduction

Suppose X is a complex Banach space and write $B(X)$ for the Banach algebra of bounded linear operators on X, X^* for the dual space of X, and $T^* \in B(X^*)$ for the dual operator of T. For $T \in B(X)$ write

$$\alpha(T) = \dim T^{-1}(0) \text{ and } \beta(T) = \operatorname{codim} T(X).$$

Thus $\alpha(T)$ and $\beta(T)$ will be either a nonnegative integer or ∞. We recall ([2], [3], [4]) that $T \in B(X)$ is called *upper semi-Fredholm* if

$$T \text{ has a closd range and } \alpha(T) < \infty$$

and is called *lower semi-Fredholm* if

$$T \text{ has a closed range and } \beta(T) < \infty.$$

If $T \in B(X)$ is either upper or lower semi-Fredholm it is called a *semi-Fredholm* operator. Write

$$\begin{aligned} \Psi_+(X) = \{T \in B(X) \ :\ & T(X) \text{ is closed} \\ & \text{and } \alpha(T-\lambda) \text{ is constant for } 0 < |\lambda| < \varepsilon\}, \\ \Psi_-(X) = \{T \in B(X) \ :\ & T(X) \text{ is closed} \\ & \text{and } \beta(T-\lambda) \text{ is constant for } 0 < |\lambda| < \varepsilon\}, \\ \Psi_\pm(X) = \Psi_+(X) & \cup \Psi_-(X). \end{aligned}$$

Evidently, we have

$$\{T \in B(X) \ : T \text{ is semi-Fredholm}\} \subseteq \Psi_\pm(X).$$

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West ([7]) defined a jump (essentially, due to Kato ([5])) of a semi-Fredholm operator. We now extend this concept to the case of a larger class. If $T \in \Psi_\pm(X)$ we define the *upper jump*, $j_+(T)$, and the *lower jump*, $j_-(T)$ of T by setting

$$j_+(T) = \alpha(T) - \alpha(T-\lambda)\,,\ \ 0 < |\lambda| < \varepsilon\,,\quad T \in \Psi_+(X)$$

$$j_-(T) = \beta(T) - \beta(T-\lambda)\,,\ \ 0 < |\lambda| < \varepsilon\,,\quad T \in \Psi_-(X)$$

with the understanding that for any real number r,

$$\infty - r = \infty$$

and that $j_+(T) = 0$ ($j_-(T) = 0$, resp.) whenever $\alpha(T)$ ($\beta(T)$, resp.) and $\alpha(T-\lambda)$ ($\beta(T-\lambda)$, resp.) are both ∞. Note that if $T \in B(X)$ is Fredholm then the continuity of the index ensures that $j_+(T) = j_-(T) < \infty$. We also recall ([1], [3], [4]) that $T \in B(X)$ is said to be *regular* if there is $T' \in B(X)$ for which $T = TT'T$. It is known that if $T \in B(X)$ is regular then $T^{-1}(0)$ and $T(X)$ are complemented in X. Thus

$$T \text{ is Fredholm} \implies T \text{ is regular.}$$

Kato's decomposition theorem ([5, Theorem 4]) says that if $T \in B(X)$ is semi-Fredholm then $T = T_1 \oplus T_2$, where T_1 is a nilpotent and $j_+(T_2) = 0$ (or $j_-(T_2) = 0$). In this paper, we shall show that if $T \in \Psi_\pm(X)$ is regular with some additional conditions then Kato's decomposition allows for T.

2. Main results

If $T \in B(X)$ then its *hyperrange* and *hyperkernel* are defined by subspaces([3], [6], [7], [8])

$$T^\infty(X) = \bigcap_{n=1}^{\infty} T^n(X)$$

and

$$T^{-\infty}(0) = \bigcup_{n=1}^{\infty} (T^n)^{-1}(0).$$

If $T \in B(X)$ is semi-Fredholm, then $T^\infty(X)$ is closed in X. If we define

$$\operatorname{comm}(T) = \{S \in B(X) \ : \ ST = TS\}$$

for the *commutant* of T, write

$$U^\wedge \ : \ T^\infty(X) \longrightarrow T^\infty(X)$$

for the operator induced by $U \in \operatorname{comm}(T)$. It is well known ([7, Propositions 2.1 and 1.7]) that

(2.1)

$$T \text{ is upper semi-Fredholm} \implies T^\wedge \text{ is Fredholm and } \beta(T^\wedge) = 0.$$

The following theorem is an improvement of both [4,(7.8.3.4)] and [7, Proposition 1.2]:

LEMMA 1. *If $S \in B(X)$ is invertible and commutes with $T \in B(X)$ then*

$$(T - S)^{-\infty}(0) \subseteq T^\infty(X) \tag{2.2}$$

and

$$T^{-\infty}(0) \subseteq (T - S)^\infty(X). \tag{2.3}$$

Proof. Towards (2.2) suppose $x \in (T-S)^{-\infty}(0)$. Then $(T-S)^m x = 0$ for some $m \in N$. Thus

$$\sum_{k=0}^{m} {}_mC_k \, T^k(-S)^{m-k}(x) = 0$$

$$\implies (TU)(x) = (-1)^{m+1} S^m(x) \qquad \text{with } U = \sum_{k=1}^{m} {}_mC_k \, T^{k-1}(-S)^{m-k}$$

$$\implies x = (-1)^{m+1} S^{-m}(TU)(x).$$

Observe that U commutes with T and S. Thus, for each $n \in N$,

$$\begin{aligned} x = (-1)^{m+1} S^{-m}(TU)(x) &= (-1)^{n(m+1)} S^{-mn} T^n U^n(x) \\ &= T^n (-1)^{n(m+1)} S^{-mn} U^n(x) \\ &\in T^n(X), \end{aligned}$$

which implies that $x \in T^{\infty}(X)$. This gives (2.2)

Towards (2.3) suppose that $x \in T^{-\infty}(0)$. Then $T^{n+1}(x) = 0$ for some $n \in N$.
Thus, for each $m \in N$,

$$\begin{aligned} T^{n+1}(x) = 0 &\implies x = (S-T)S^{-1}(I + S^{-1}T + \cdots + S^{-n}T^n)(x) \\ &\implies x = (S-T)^m S^{-m}(I + S^{-1}T + \cdots + S^{-n}T^n)^m(x) \\ &\qquad \in (T-S)^m(X), \end{aligned}$$

which implies that $x \in (T-S)^{\infty}(X)$. This gives (2.3).

THEOREM 2. *If $T \in \Psi_+(X)$ has a finite dimensional intersection $T^{-1}(0) \cap T^k(X)$ for some $k \in N$ then*

(2.4) $$j_+(T) = 0 \text{ if and only if } T^{-\infty}(0) \subseteq T^{\infty}(X).$$

Proof. Suppose that $T \in \Psi_+(X)$. If $j_+(T) = 0$ we claim that
(2.5)
$$\alpha(T^\wedge) \leq \alpha(T) = \alpha(T-\lambda) = \alpha(T^\wedge - \lambda) \leq \alpha(T^\wedge), \text{ for } 0 < |\lambda| < \varepsilon.$$

Indeed, the first inequality is evident, the second equality comes from the assumption, the third equality comes from (2.2), and the last inequality comes from an argument of Kato([5, Theorem 1]). Thus (2.5) gives

$$\alpha(T) = \alpha(T^\wedge).$$

Thus, since, by assumption, $\dim T^{-1}(0) = \dim T^{-1}(0) \cap T^{\infty}(X) < \infty$, it follows that $T^{-1}(0) \subseteq T^{\infty}(X)$, and hence $T^{-\infty}(0) \subseteq T^{\infty}(X)$.

Conversely, suppose that $T^{-\infty}(0) \subseteq T^{\infty}(X)$. Then

$$T^{-1}(0) = T^{-1}(0) \cap T^{\infty}(X) \subseteq T^{-1}(0) \cap T^k(X)$$

is finite dimensional; thus T is upper semi-Fredholm. By (2.1), $T^\wedge$ is Fredholm and $\beta(T^\wedge) = 0$. Thus we have

$$\alpha(T) = \alpha(T^\wedge) = \alpha(T^\wedge - \lambda) = \alpha(T - \lambda) \quad \text{for } 0 < |\lambda| < \varepsilon,$$

which says that $j_+(T) = 0$.

We also have the dual result:

THEOREM 3. *If $T \in \Psi_-(X)$ has a finite dimensional intersection $(T^k)^{-1}(0)^\perp \cap T(X)^\perp$ for some $k \in N$ then*

$$j_-(T) = 0 \text{ if and only if } T^{-\infty}(0) \subseteq T^\infty(X).$$

Proof. Suppose that $T \in \Psi_-(X)$. Remembering that

$$T \text{ has a closed range } \iff T^* \text{ has a closed range}$$

and $\beta(T) = \alpha(T^*)$ we have

$$T^* \in \Psi_+(X^*) \text{ and } j_-(T) = 0 \iff j_+(T^*) = 0.$$

Furthermore, a direct calculation shows that

$$\dim\,[(T^*)^{-1}(0) \cap (T^k)^*(X^*)] \;=\; \dim\,[(T(X)^\perp \cap (T^k)^{-1}(0)^\perp] < \infty$$

and

$$(T^*)^{-\infty}(0) \subseteq T^*(X^*) \iff T^{-\infty}(0) \subseteq T^\infty(X).$$

Now applying Theorem 2 gives the required result.

Our main theorem is an extension of Kato's decomposition theorem :

THEOREM 4. *If $T \in \Psi_+(X)$ satisfies that*

$$T^{-1}(0) \text{ and } T^{-1}(0) + T(X) \text{ are complemented,}$$
$$T^{-1}(0) \cap T(X) \text{ is finite dimensional}$$

then we have a decomposition

$$T = T_1 \oplus T_2,$$

where T_1 is nilpotent and T_2 is upper semi-Fredholm with $j_+(T_2) = 0$.

Proof. If $j_+(T) = 0$, then by (2.4), $T^{-1}(0) \subseteq T^\infty(X)$; thus our assumption says that T is upper semi-Fredholm. Thus there is nothing to prove.

If $j_+(T) \neq 0$, then there is a smallest integer v such that

$$T^{-1}(0) \subseteq T^v(X), \text{ but } T^{-1}(0) \not\subseteq T^{v+1}(X).$$

If $v \geq 1$ then, by assumption,

$$T^{-1}(0) = T^{-1}(0) \cap T^v(X) \subseteq T^{-1}(0) \cap T(X)$$

is finite dimensional and hence T is upper semi-Fredholm; in this case, an argument of West ([8, Theorem 7]) gives the required result. Now suppose

$$T^{-1}(0) \not\subseteq T(X).$$

By assumption, we can find closed subspaces Y, Z and W for which

$$\begin{aligned} T^{-1}(0) &= Y \oplus T^{-1}(0) \cap T(X) \\ T(X) &= T^{-1}(0) \cap T(X) \oplus Z \end{aligned}$$

and

$$X = T^{-1}(0) \oplus Z \oplus W.$$

Thus there are continuous projections $P \in B(X)$ and $Q \in B(T^{-1}(0))$ for which

$$P(X) = T^{-1}(0) \text{ and } P^{-1}(0) = Z \oplus W$$

and

$$Q(T^{-1}(0)) = Y \text{ and } Q^{-1}(0) = T^{-1}(0) \cap T(X).$$

Then we have

$$QP = (QP)^2,$$

so that QP is a continuous projection on X with range Y. Further,

$$\begin{aligned} T(QP(X)) &= T(Y) = \{0\}, \\ QP(TX) &= QP(Z \oplus T^{-1}(0) \cap T(X)) \\ &= Q(T^{-1}(0) \cap T(X)) = \{0\}. \end{aligned}$$

Thus T is reduced by the decomposition $X = (QP)(X) \oplus (QP)^{-1}(0)$. We write

$$T = S_1 \oplus S_2,$$

where $S_1 = T|_{(QP)(X)}$ and $S_2 = T|_{(QP)^{-1}(0)}$. Note that $S_1 = 0$. Since S_2 has a closed range and

$$S_2^{-1}(0) = (QP)^{-1}(0) \cap T^{-1}(0) = T^{-1}(0) \cap T(X)$$

is finite dimensional, it follows that S_2 is upper semi-Fredholm. Again, an argument of West ([8, Theorem 7]) gives

$$S_2 = R_1 \oplus R_2,$$

where R_1 is nilpotent and R_2 is upper semi-Fredholm with $j_+(R_2) = 0$. Thus we have

$$T = T_1 \oplus T_2 \text{ with } T_1 = O \oplus R_1 \text{ and } T_2 = R_2;$$

thus T_1 is also nilpotent. This completes the proof.

We conclude with the dual result :

THEOREM 5. *If $T \in \Psi_-(X)$ satisfies that*

$T(X)$ *is complemented,*

$T(X) + T^{-1}(0)$ *is finite codimensional*

then

$$T = T_1 \oplus T_2,$$

where T_1 is nilpotent and T_2 is lower semi-Fredholm with $j_-(T_2) = 0$.

Proof. By assumption, we can find closed subspaces Y and W for which

$$X = (T^{-1}(0) + T(X)) \oplus W$$

and

$$T^{-1}(0) + T(X) = T(X) \oplus Y \quad \text{with } Y \subseteq T^{-1}(0),$$

and hence

$$X = T(X) \oplus Y \oplus W,$$

Thus there are continuous projections $P \in B(X)$ and $Q \in B(T^{-1}(0) + T(X))$ for which

$$P(X) = T^{-1}(0) + T(X) \text{ and } P^{-1}(0) = W$$

and

$$Q(T^{-1}(0) + T(X)) = Y \text{ and } Q^{-1}(0) = T(X).$$

Then we have

$$QP = (QP)^2,$$

so that QP is a continuous projection on X with range Y. Further,

$$\begin{aligned} T(QP(X)) &= T(Y) = \{0\}, \\ QP(TX) &= Q(T(X)) = \{0\}. \end{aligned}$$

Thus T is reduced by the decomposition $X = (QP)(X) \oplus (QP)^{-1}(0)$. We write

$$T = S_1 \oplus S_2,$$

where $S_1 = T|_{(QP)(X)}$ and $S_2 = T|_{(QP)^{-1}(0)}$. Note that $S_1 = 0$. Since $(QP)^{-1}(0) = T(X) \oplus W$, S_2 has a finite codimensional range. It thus follows that S_2 is lower semi-Fredholm. Again, an argument of West ([8, Theorem 7]) gives

$$S_2 = R_1 \oplus R_2,$$

where R_1 is nilpotent and R_2 is lower semi-Fredholm with $j_-(R_2) = 0$. Thus we have

$$T = T_1 \oplus T_2 \text{ with } T_1 = O \oplus R_1 \text{ and } T_2 = R_2;$$

thus T_1 is also nilpotent. This completes the proof.

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