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SOLUTIONS OF NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS IN L^p SPACES

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1. Introduction

Let X be a real Banach space with norm $\|\cdot\|$. Let $T > 0$, $r \geq 0$ be fixed constants. We denote by L^p the usual $L^p(-r, 0; X)$ with norm $\|\cdot\|_p$ for $1 \leq p < \infty$. Our object is to study the existence of solutions of nonlinear functional evolution equations of the type

$$\text{(FDE)} \qquad \begin{cases} x'(t) + A(t)x(t) = G(t, x_t), & 0 \leq t \leq T, \\ x_0 = \phi. \end{cases}$$

The symbol x_t denotes the function $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$.

For (FDE) we assume the followings :

(A1) There exists $\alpha \in \mathcal{R}$ such that for each $t \in [0, T]$, $A(t) + \alpha I$ is accretive and $R(I + \lambda A(t)) = X$ for $0 < \lambda < \lambda_0 = 1/\max(0, \alpha)$.

(A2) There exist a continuous function $h : [0, T] \to X$ which is of bounded variation on $[0, T]$ and a continuous nondecreasing function $L_1 : [0, \infty) \to [0, \infty)$ such that

$$\|A_\lambda(t)x - A_\lambda(s)x\| \leq \|h(t) - h(s)\| L_1(\|x\|)(1 + \|A_\lambda(s)x\|)$$

for $0 < \lambda \leq \lambda_0$, $0 \leq s, t \leq T$, where $A_\lambda(t)$ is the Yosida approximant of $A(t)$.

(A3) There is a constant $\beta > 0$ such that

$$\|G(t, \phi) - G(t, \psi)\| \leq \beta \|\phi - \psi\|_p$$

for $\phi, \psi \in L^p$ and $t \in [0, T]$.

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(A4) There are a continuous function $k : [0,T] \to X$ which is of bounded variation on $[0,T]$ and a continuous function $L_2 : [0,\infty) \to [0,\infty)$ such that for $0 \le s,t \le T$ and $\psi \in L^p$

$$\|G(t,\psi) - G(s,\psi)\| \le \|k(t) - k(s)\| L_2(\|\psi\|_p).$$

Many authors have been studied for last two decades the type of (FDE) with various settings on space X, operators $A(t)$, and initial function ϕ (cf. Dyson and Villella-Bressan [2, 3, 4], Kartsatos and Parrott [7, 8], and Webb [13, 14]). Recently, Kartsatos and Parrott [7], Tanaka [10] have proved the existence of generalized solutions of (FDE) assuming (A1)- (A4) with Lipschitzian ϕ. To improve on the initial function, we take an approach which has been used by Dyson and Villella-Bressan [3], Webb [14] except showing the existence of Discrete Scheme (DS)-limit solution to project.

This paper consists of two parts. First we recall the basic nonlinear operator theory that we use later. Also we define an operator in a product space to get a nonautonomous evolution equation. Then, we show the existence of generalized solutions of (FDE) by projecting solutions in the product space to X. We also discuss the generalized domain briefly.

2. Preliminaries

Let Y be a real Banach space with its dual Y^* and $\langle y, z\rangle$ denote the evaluation $z(y)$ for $y \in Y$ and $z \in Y^*$. Define $J_Y y = \{y^* \in Y^* : \langle y, y^*\rangle = \|y\|^2 = \|y^*\|^2\}$. ($J_Y y$ is nonempty for each $y \in Y$ by the Hahn-Banach theorem.) The mapping J_Y is called the duality mapping of Y. An operator $T : D(T) \subset Y \to Y$ is accretive if for each $\lambda > 0$ and $x, y \in D(A)$ $\|x - y\| \le \|x - y + \lambda(Tx - Ty)\|$. Equivalently, (see Kato [5]) T is accretive if and only if for every $x, y \in Y$ there is $j \in J_Y(x-y)$ such that $\langle Tx - Ty, j \rangle \ge 0$. An operator $T : D(T) \subset Y \to Y$ is said to be $\mathcal{A}(\omega)$ if for each $\lambda > 0$ with $\lambda\omega < 1$ and $x, y \in D(T)$

$$\|x - y + \lambda(Tx - Ty)\| \ge (1 - \lambda\omega)\|x - y\|. \tag{1}$$

Note that $T + \omega I$ is accretive if and only if $T \in \mathcal{A}(\omega)$. Also (1) implies that $(I + \lambda T)^{-1}$ exists on $R(I + \lambda T)$ and is Lipschitz continuous with constant $(1 - \lambda\omega)^{-1}$ on $R(I + \lambda T)$.

The resolvents and Yosida approximants of T, J_λ and T_λ, are defined by $J_\lambda y = (I + \lambda T)^{-1} y$ and $T_\lambda y = \frac{1}{\lambda}(I - J_\lambda)y$, respectively. It is readily verified that $T_\lambda y = TJ_\lambda y$. We define $|Ty|$ by $|Ty| = \lim_{\lambda \downarrow 0} \|T_\lambda y\|$. If $T \in \mathcal{A}(\omega)$ and $R(I + \lambda T) = Y$ for all $0 < \lambda \leq \lambda_0$, then the limit exists even though it may be infinite. For such T we define the generalized domain of T $\hat{D}(T) = \{y \in Y : |Ty| < \infty\}$. Then $D(T) \subset \hat{D}(T)$. For other properties of J_λ, T_λ, and $|Ty|$ which hold in a general Banach space Y, we refer the reader to Crandall and Pazy [1].

We recall that

$$\langle y, x \rangle_+ = \lim_{h \to 0+} \frac{\|x + hy\| - \|x\|}{h}.$$

An integral solution of (FDE) is a function $x : [-r, T] \to X$ such that $x_0 = \phi$, x is continuous and satisfies the inequality

$$\|x(t) - y\| - \|x(s) - y\| \leq \int_s^t (< -A(\tau)y + G(\tau, x_\tau), x(\tau) - y\rangle_+ + \alpha\|x(\tau) - y\|) d\tau$$

for all $y \in D(A(r))$, $r \in [0, T]$, and $0 \leq s \leq \tau \leq t \leq T$.

Let $Y = L^p \times X$ be a Banach space with norm

$$\|\{\phi, h\}\|_Y = \left(\int_{-r}^0 \|\phi(\theta)\|^p d\theta + \|h\|^p \right)^{\frac{1}{p}}$$

for every $\{\phi, h\} \in Y$.

Due to Webb [14], the duality mapping J_Y of Y is given by following:

PROPOSITION 1. *If $\{\phi, h\} \in Y$, then $j \in J_Y(\{\phi, h\})$ where j is defined by*

$$\langle \{\psi, k\}, j \rangle = \|\{\phi, h\}\|_Y^{2-p} \cdot \left(\int_{-r}^0 < \psi(\theta), \phi^*(\theta) \rangle \|\phi(\theta)\|^{p-2} d\theta + \langle k, h^* \rangle \|h\|^{p-2} \right)$$

for all $\{\psi, k\} \in Y$, $\phi^* \in J_{L^p}(\phi)$ *and* $h^* \in J_X(h)$.

We define a family of nonlinear operators, for $0 \leq t \leq T$, $B(t) : D(B(t)) \subset Y \to Y$ by

$$B(t)\{\phi, h\} = \{-\phi', A(t)h - G(t, \phi)\}, \tag{3}$$
$$D(B(t)) = \{\{\phi, h\} \in Y : \phi \in W^{1,p}(-r, 0; X), \phi(0) = h \in D(A(t))\}.$$

PROPOSITION 2. (Parrott [10], Tanaka [14]) *Let* $\{A(t) : t \in [0,T]\}$ *satisfy* (A1), *and suppose* $G : [0,T] \times L^p \to X$ *satisfies* (A3). *If* $\{B(t) : t \in [0,T]\}$ *is a family of operators in* Y *defined in* (3), *then* $B(t) \in \mathcal{A}(\gamma)$ *for* $\gamma = \max(0, \alpha + 1/p) + \beta$ *and* $R(I + \lambda B(t)) = Y$ *for sufficiently small* $\lambda > 0$.

PROPOSITION 3. (Tanaka [14]) *Let* $A(t)$ *and* $G(t, \cdot)$, $0 \leq t \leq T$, *satisfy* (A1)–(A4). *Then there exist a continuous function* $f : [0,T] \to Y$ *which is of bounded variation and a nondecreasing continuous function* $L : [0, \infty) \to [0, \infty)$ *such that*

$$\|B_\lambda(t)u - B_\lambda(s)u\|_Y \leq \|f(t) - f(s)\|_Y L(\|u\|_Y)(1 + \|B_\lambda(s)u\|_Y)$$

for each $0 \leq s, t \leq T$, $u \in Y$ *and sufficiently small* $\lambda > 0$.

THEOREM 1. (Tanaka [14]) *Let* $A(t)$ *and* $G(t, \cdot)$, $0 \leq t \leq T$, *satisfy* (A1)-(A4). *Then, a family of operators* $B(t)$, $0 \leq t \leq T$, *defined in* (3) *satisfies followings :*

(B1) *For each* $t \in [0,T]$, $B(t) \in \mathcal{A}(\gamma)$ *for* $\gamma = \max(0, \alpha + 1/p) + \beta$, *and* $R(I + \lambda B(t)) = Y$ *for sufficiently small* $\lambda > 0$.

(B2) *There exist a continuous function* $f : [0,T] \to Y$ *which is of bounded variation and a continuous nondecreasing function* $L : [0, \infty) \to [0, \infty)$ *such that*

$$\|B_\lambda(t)u - B_\lambda(s)u\|_Y \leq \|f(t) - f(s)\|_Y L(\|u\|_Y)(1 + \|B_\lambda(s)u\|_Y)$$

for $0 \leq s, t \leq T$, $u \in Y$ *and sufficiently small* $\lambda > 0$.

3. Main results

We now consider a nonlinear evolution equation in $Y = L^p \times X$ of the form

$$\text{(EE)} \qquad u'(t) + B(t)u(t) = 0, \quad 0 \le t \le T, \quad u(0) = u_0,$$

where the operator $B(t)$ is defined in (2) satisfying (B1)-(B2) and for some $u_0 \in \overline{D(B(t))}$. First we note that for each $t \in [0,T]$

$$D(A(t)) \subset \hat{D}(A(t)) \quad \text{and} \quad D(B(t)) \subset \hat{D}(B(t)).$$

Moreover, the generalized domains $\hat{D}(A(t))$ and $\hat{D}(B(t))$ are constants since $A(t)$ and $B(t)$ satisfy the inequalities (A2) and (B2) (cf. Evans [5]). We denote by $\hat{D}_A$ and $\hat{D}_B$, respectively.

Let there exist a sequence of partitions $P_n = \{0 = t_0^n < t_1^n < \cdots < t_{N(n)}^n = T\}$ and a sequence $\{u_j^n\}$, $j = 0, 1, \ldots, N(n)$ of elements of Y such that

(1) $\frac{u_j^n - u_{j-1}^n}{t_j^n - t_{j-1}^n} + B(t_j^n)u_j^n = 0$, $j = 1, 2, \ldots, N(n)$, $n = 1, 2, \ldots$.

(2) $\lim_{n\to\infty} \max_{1 \le j \le N(n)} (t_j^n - t_{j-1}^n) = 0$.

(3) $u_0^n = u_0$.

The step function u_n on $[0,T]$ defined by

$$u_n(t) = \begin{cases} u_0, & t = 0 \\ u_j^n, & t \in (t_{j-1}^n, t_j^n],\ j = 1, 2, \ldots, N(n), \end{cases}$$

is called DS-approximate solution of (EE).

If DS-approximate solution u_n converges to some continuous function u uniformly on $[0,T]$, we call it DS-limit solution of (EE).

The next theorem shows the existence of DS-limit solution of (EE).

THEOREM 2. *Let $B(t)$, $0 \le t \le T$, satisfy* (B1)-(B2). *Then for $u_0 \in \overline{D_B}$ there exists a DS-limit solution of* (EE).

Proof. Suppose we have two DS-approximate solutions $u_n(t)$, $v_m(t)$ on $[0,T]$ defined by sequences $\{t_j^n\}$, $\{u_j^n\}$, $j = 0, 1, \ldots, N(n)$ and $\{\hat{t}_k^m\}$, $\{v_k^m\}$, $k = 0, 1, 2, \ldots, N(m)$ with u_0^n, $v_0^m \in D(B(0))$ satisfying

$$\lim_{n\to\infty} u_0^n = \lim_{m\to\infty} v_0^m = u_0.$$

Let $\tilde{u} \in D(B(r))$ for some $r \in [0,T]$. To simplify notation, we denote by $\|\cdot\|$ norm on Y in this proof. Most of the proof is the same way which Evans ([6]) and Pavel ([11]) have used. First, we show $\|u_j^n\| \le M_1$, where M_1 is independent of n and j. Set $\gamma_j = t_j^n - t_{j-1}^n$, $\delta_k = \hat{t}_k^m - \hat{t}_{k-1}^m$, and $\sigma_{j,k} = \delta_k\gamma_j/(\gamma_j+\delta_k)$. Let $d_n = \max_{1\le j\le N(n)} \gamma_j$ be such that $\gamma d_n < 1/2$. We estimate $\|u_j^n - \tilde{u}\|$. Indeed,

$$\begin{aligned}
&\|u_j^n - \tilde{u}\| \le \|J_{\gamma_j}^B(t_j^n)u_{j-1}^n - J_{\gamma_j}^B(t_j^n)\tilde{u}\| + \|J_{\gamma_j}^B(t_j^n)\tilde{u} - \tilde{u}\| \\
&\le (1-\gamma_j\gamma)^{-1}\{\|u_{j-1}^n - \tilde{u}\| + \gamma_j|B(r)\tilde{u}| \\
&\quad + \gamma_j\|f(t_j^n) - f(r)\|L(\|\tilde{u}\|)(1+|B(r)\tilde{u}|)\} \\
&\le (1-\gamma_j\gamma)^{-1}(1-\gamma_{j-1}\gamma)^{-1}\{\|u_{j-2}^n - \tilde{u}\| + (\gamma_j+\gamma_{j-1})|B(r)\tilde{u}| \\
&\quad + (\gamma_j\|f(t_j^n) - f(r)\| + \gamma_{j-1}\|f(t_{j-1}^n - f(r)\|)L(\|\tilde{u}\|)(1+|B(r)\tilde{u}|)\}.
\end{aligned}$$

Continuing this process,

$$\begin{aligned}
\|u_j^n - \tilde{u}\| = \prod_{i=1}^{n}(1-\gamma_i\gamma)^{-1}\{&\|u_0^n - \tilde{u}\| + t_j^n|B(r)\tilde{u}| \\
&+ \sum_{i=1}^{j}\gamma_i\|f(t_i^n) - f(r)\|L(\|\tilde{u}\|)(1+|B(r)\tilde{u}|)\}.
\end{aligned}$$

Using $(1-\gamma_j\gamma)^{-1} \le e^{2t_j^n\gamma} \le e^{2T\gamma}$, we have

$$\begin{aligned}
\|u_j^n\| &= \|\tilde{u}\|e^{2T\gamma}\{\|u_0^n - \tilde{u}\| + t_j^n|B(r)\tilde{u}| \\
&\qquad + \sum_{i=1}^{j}\gamma_i\|f(t_i^n) - f(r)\|L(\|\tilde{u}\|)(1+|B(r)\tilde{u}|)\} \\
&\le M_1.
\end{aligned}$$

Next, we show $\|\frac{u_j^n - u_{j-1}^n}{\gamma_j}\| \le M_2$, where M_2 is independent of n and j. Set $a_j = |B(t_j^n)u_{j-1}^n|$ and $b_j = \|f(t_j^n) - f(t_{j-1}^n)\|L(\|u_{j-1}^n\|)$. Then, since

$$\begin{aligned}
|B(t_j^n)u_{j-1}^n| \le\ & |B(t_{j-1}^n)u_{j-1}^n| \\
&+ \|f(t_j^n) - f(t_{j-1}^n)\|L(\|u_{j-1}^n\|)(1+|B(t_{j-1}^n)u_{j-1}^n|),
\end{aligned}$$

we have

$$a_j \le (1-\gamma_j\gamma)^{-1}(1+b_j)a_{j-1}+b_j.$$

Thus

$$\begin{aligned}\|\frac{u_j^n-u_{j-1}^n}{\gamma_j}\| &\\ &\le (1-\gamma_j\gamma)^{-1}a_j\\ &\le (1-\gamma_j\gamma)^{-1}(1-\gamma_{j-1}\gamma)^{-1}(1-\gamma_{j-2}\gamma)^{-1}(1+b_j)(1+b_{j-1})a_{j-2}\\ &\quad +(1-\gamma_j\gamma)^{-1}(1-\gamma_{j-1}\gamma)^{-1}(1+b_j)b_{j-1}+b_{j-1}+(1-\gamma_j\gamma)^{-1}b_j.\end{aligned}$$

Continuing this process with $\prod_{l=i}^{j}(1+b_l) \le \exp(\sum_{l=2}^{j} b_l)$, we get

$$\|\frac{u_j^n-u_{j-1}^n}{\gamma_j}\| \le (a_1+\sum_{i=2}^{j} b_i)\exp(2\gamma t_j^n)\exp(\sum_{i=2}^{j} b_i).$$

Therefore,

$$\begin{aligned}\|\frac{u_j^n-u_{j-1}^n}{\gamma_j}\| &\le (|B(t_1^n)u_0^n|+L(M_1))\exp(2\gamma T)\exp(L(M_1)\ \mathrm{Var}\ f)\\ &\le \big(B(t_0^n)u_0^n|+\|f(t_1^n)-f(t_0^n)\|L(\|u_0^n\|)\\ &\quad \cdot(1+|B(t_0^n)u_0^n|)+L(M_1)\big)\exp(2\gamma T)\exp(L(M_1)\ \mathrm{Var}\ f)\\ &\le M_2.\end{aligned}$$

Using the above two results, we take the same steps in the proof of Lemma 5.1 (Evans [6]) to get the following result :

$$\begin{aligned}(1-\sigma_{j,k})\|u_j^n-v_k^m\| \le \frac{\delta_k}{\gamma_j+\delta_k}\|u_{j-1}^n-v_k^m\|+\frac{\gamma_j}{\gamma_j+\delta_k}\|u_j^n-v_{k-1}^m\|\\ +\sigma_{j,k}\|f(t_j^n)-f(t_k^m)\|L(M_1)(1+M_2)\end{aligned}$$

for $1 \le j \le N(n)$, and $1 \le k \le N(m)$. Now, we introduce the concept of the "modulus of continuity" of f to follow Pavel [11]. We apply the method of Lemma 2.3 of [11]. Then we get

$$\begin{aligned} w_{j,k}\|u_j^n - v_k^m\| \le \|u_0^n - u_0\| &+ \|v_0^m - u_0\| + 2\|\tilde{u} - u_0\| \\ &+ C_{j,k}(B(r)\tilde{u}| + K\rho(T)(1 + |B(r)\tilde{u}|)) \\ &+ K\hat{t}_k^m(\frac{1}{c}\rho(T)C_{j,k} + \rho(\sigma)), \end{aligned}$$

where $C_{j,k} = ((t_j^n - \hat{t}_k^m)^2 + d_n t_j^n + \hat{d}_m \hat{t}_k^m)^{1/2}$, ρ is the modulus of continuity of f, and $w_{j,k} = \prod_{i=1}^{j}(1 - \gamma_i\gamma)\prod_{i=1}^{k}(1 - \delta_i\gamma)$. Let

$$u_n(t) = \begin{cases} u_0^n, & t = 0, \\ u_j^n, & t \in (t_{j-1}^n, t_j^n], j = 1, 2, \dots, N(n). \end{cases}$$

Then, $\lim_{n,m\to\infty}\|u_n(t) - u_m(t)\| = 0$ uniformly on $[0, T]$. Define $u(t) = \lim_{n\to\infty} u_n(t)$. Also we follow the same steps in Theorem 3.1 (Pavel [12]) to show that $u(t)$ is continuous. Therefore, $u(t)$ is a DS-limit solution of (EE).

DEFINITION 1. Let π_1, π_2 are projections from $Y = L^p \times X$ into L^p and X, respectively. A function $u : [0, T] \to Y$ is called a translation if $\pi_1 u(t) = x_t$ where $x(t)$ is defined by

$$x(t) = \begin{cases} \phi(t), & -r \le t < 0, \\ \pi_2 u(t), & 0 \le t \le T. \end{cases}$$

By Plant [13], a DS-limit solution $u(t)$ of (EE) is a translation.

Consider the existence of a DS-limit solution of (FDE). Let $\phi \in L^p$ with $\phi(0) \in \overline{D_A}$. When $p = 1$, since $C = C([-r, 0]; X)$ is dense in $L^1 = L^1(-r, 0; X)$ with respect to L^1-norm, for every $\epsilon > 0$ there exists $\tilde{\phi} \in C$ such that $\|\tilde{\phi} - \phi\|_1 < \epsilon$. For $\tilde{\phi} \in C$, there exists $\phi_n \in C^\infty = C^\infty([-r, 0]; X) \subset W^{1,1}$ such that $\lim_{n\to\infty} \phi_n = \tilde{\phi}$ with respect to L^1–norm. Thus $\lim_{n\to\infty} \phi_n = \phi$ with respect to the L^1-norm.

When $1 < p < \infty$, for a continuous function $\phi \in L^p$ with $\phi(0) \in \overline{D_A}$, there exists $\phi_n \in C^\infty \subset W^{1,1}$ such that $\lim_{n\to\infty} \phi_n = \phi$ with respect to L^p-norm.

Let us define

$$\psi_n(\theta) = \begin{cases} \phi_n(\theta), & -r \le \theta < 0, \\ h_n, & \theta = 0, \end{cases}$$

where $h_n \in D(A(t))$ such that $\lim_{n\to\infty} h_n = \phi(0)$ in X for $\phi(0) \in \overline{D_A}$. Then $\psi_n \in W^{1,1}$ and $\psi_n(0) \in D(A(t))$. Thus $\{\psi_n, \psi_n(0)\} \in D(B(t))$. Hence $\{\phi, \phi(0)\} \in \overline{D_B}$. Putting $u_0 = \{\phi, \phi(0)\}$, by Theorem 2, there exists a DS-limit solution $u(t)$ of (EE) i.e., there exist a sequence of partitions $P_n = \{0 = t_0^n < t_1^n < \cdots < t_{N(n)}^n = T\}$ and a sequence $\{u_j^n\}$, $j = 0, 1, \ldots, N(n)$, of elements of Y such that

$$\lim_{n\to\infty} u_n(t) = u(t) \quad \text{where} \quad u_n(t) = \begin{cases} u_0^n, & t = 0, \\ u_j^n, & t \in (t_{j-1}^n, t_j^n]. \end{cases}$$

Here,

$$\text{(4)} \qquad \begin{aligned} &\frac{u_j^n - u_{j-1}^n}{t_j^n - t_{j-1}^n} + B(t_j^n)u_j^n \ni 0,\ j = 1, 2, \ldots, N(n),\ n = 1, 2, \ldots, \\ &\text{with} \quad u_j^n = \{\phi_j^n, h_j^n\}, \quad u_0^n = \{\psi_n, \psi_n(0)\} \in D(B(0)), \\ &\lim_{n\to\infty} u_0^n = u_0, \quad \lim_{n\to\infty} \max_{1\le j\le N(n)} (t_j^n - t_{j-1}^n) = 0. \end{aligned}$$

If we project (4) into X, we have

$$\frac{h_j^n - h_{j-1}^n}{t_j^n - t_{j-1}^n} + A(t_j^n)h_j^n \ni G(t_j^n, \phi_j^n).$$

Define

$$x_n(t) = \begin{cases} \psi_n(t), & -r \le t \le 0, \\ h_j^n, & t \in (t_{j-1}^n, t_j^n], \quad j = 1, 2, \ldots, N(n). \end{cases}$$

Since $\lim_{n\to\infty} u_n(t) = u(t)$ and $h_j^n = \pi_2(u_n(t))$, $\lim_{n\to\infty} x_n(t) = x(t)$, where

$$x(t) = \begin{cases} \phi(t), & -r \le t \le 0, \\ \pi_2(u(t)), & 0 \le t \le T. \end{cases}$$

Since $u(t)$ is continuous, $x(t)$ is also continuous on $[-r, T]$. Then $x(t)$ is a DS-limit solution of (FDE).

THEOREM 3. *Let* (A1)–(A4) *be satisfied. For every* $\phi \in L^1$ *or for every continuous* $\phi \in L^p$ $(1 < p < \infty)$, *if* $\phi(0) \in \overline{D_A}$, *then there exists a* DS-*limit solution of* (FDE).

To investigate the relation between a DS-limit solution of (EE) and an integral solution of (FDE), we use the concept of translation. For an integral solution of (FDE), we have the similar result of Dyson and Villella-Bressan [2] with the following stronger conditions than (A2).

(A2)′ For single valued $A(t)$, $0 \leq t \leq T$, there exist a continuous function $h : [0, T] \to X$ which is of bounded variation and a continuous nondecreasing function $L_1 : [0, \infty) \to [0, \infty)$ such that

$$\|A(t)x - A(s)x\| \leq \|h(t) - h(s)\| L_1(\|x\|)(1 + \|A(s)x\|)$$

for $x \in D_A$.

THEOREM 4. *Assume that* $A(t) : D(A(t)) = D_A \to X$. *Let* (A1)–(A4) *be satisfied with* X^* *uniformly convex or let* (A1) (A2)′ (A3) (A4) *be satisfied. Then, when* $p = 1$, *for every* $\phi \in L^1$ *with* $\phi(0) \in \overline{D_A}$, *and when* $1 < p < \infty$, *for every continuous function* $\phi \in L^p$ *with* $\phi(0) \in \overline{D_A}$, *there exists an integral solution of* (FDE).

Proof. We note that (A2) with uniformly convex dual X^* implies (A2)′ and (A2)′ implies that (A2). As in the proof of Theorem 3, we have a DS-approximate solution $u_n(t)$ in Y satisfying (4). Since

$$\frac{u_j^n - u_{j-1}^n}{t_j^n - t_{j-1}^n} + B(t_j^n)u_j^n = 0, \quad j = 1, 2, \ldots, N(n), \quad n = 1, 2, \ldots,$$

with $u_j^n = \{\phi_j^n, h_j^n\}$, if we project into X, we have

$$\frac{h_j^n - h_{j-1}^n}{t_j^n - t_{j-1}^n} + A(t_j^n)h_j^n - G(t_j^n, \phi_j^n) = 0,$$

for $j = 1, 2, \ldots, N(n)$, $n = 1, 2, \ldots$. Put $\delta_j = t_j^n - t_{j-1}^n$. Let $x \in D_A$ be arbitrary and $s \in [0, T]$. For $j^* \in J_X(h_j^n - x)$, since

$$\begin{aligned}\|h_j^n - x\|^2 &= \langle h_j^n - x, j^* \rangle \\ &= \langle h_j^n - h_{j-1}^n, j^* \rangle + \langle h_{j-1}^n - x, j^* \rangle \\ &\leq \langle h_j^n - h_{j-1}^n, j^* \rangle + \|h_{j-1}^n - x\| \cdot \|h_j^n - x\|\end{aligned}$$

we get

$$\begin{aligned}
&\|h_j^n - x\|^2 - \|h_j^n - x\| \cdot \|h_{j-1}^n - x\| \le \langle h_j^n - h_{j-1}^n, j^*\rangle \\
&= \delta_j\langle -A(t_j^n)h_j^n + G(t_j^n, \phi_j^n), j^*\rangle \\
&= -\langle A(t_j^n)h_j^n - A(t_j^n)x, j^*\rangle + \delta_j\langle A(s)x - A(t_j^n)x, j^*\rangle \\
&\quad + \delta_j\langle -A(s)x + G(t_j^n, \phi_j^n), j^*\rangle \\
&\le \delta_j\alpha\|h_j^n - x\|^2 + \delta_j\|A(t_j^n)x - A(s)x\| \cdot \|h_j^n - x\| \\
&\quad + \delta_j\langle -A(s)x + G(t_j^n, \phi_j^n), h_j^n - x\rangle_+.
\end{aligned}$$

Hence, by (A2)$'$

$$\begin{aligned}
&\|h_j^n - x\| - \|h_j^n - x\| \\
&\quad \le \delta_j\alpha\|h_j^n - x\| + \delta_j\|h(t_j^n)x - h(s)x\|L_1(\|x\|)(1 + \|A(s)x\|) \\
&\quad\quad + \delta_j\langle -A(s)x + G(t_j^n, \phi_j^n), h_j^n - x\rangle_+.
\end{aligned}$$

Iterating for $j = i+1, \ldots, k$, $(i+1 < k)$, with $C = L_1(\|x\|)(1+\|A(s)x\|)$

$$\begin{aligned}
&\|h_j^n - x\| - \|h_j^n - x\| \\
&\quad \le \sum_{j=i+1}^{k} (\alpha\|h_j^n - x\| + C\|h(t_j^n)x - h(s)x\| \\
&\quad\quad + \delta_j\langle -A(s)x + G(t_j^n, \phi_j^n), h_j^n - x\rangle_+.
\end{aligned}$$

Let $k = k_n$, $i = i_n$ be such that $t \in (t_{k_n-1}^n, t_{k_n}^n]$ and $\bar{t} \in (t_{i_n-1}^n, t_{i_n}^n]$. Set $a_n(\sigma) = t_j^n$ for $\sigma \in (t_{j-1}^n, t_j^n]$. Then

$$\begin{aligned}
&\|\pi_2(u_n(t)) - x\| - \|\pi_2(u_n(\bar{t})) - x\| \\
&\le \int_{t_{i_n}^n}^{t_{k_n}^n} \big(\langle -A(s)x + G(a_n(\sigma), \pi_1(u_n(\sigma)), \pi_2(u_n(\sigma)) - x\rangle_+ \\
&\quad + \alpha\|\pi_2(u_n(\sigma)) - x\| + C\|h(a_n(\sigma)) - h(s)\|\big)d\sigma.
\end{aligned}$$

Since $a_n(\sigma) \to \sigma$ as $n \to \infty$, passing to the limit or $n \to \infty$, we have

$$\begin{aligned}
&\|\pi_2(u(t)) - x\| - \|\pi_2(u(\bar{t})) - x\| \\
&\le \int_{\bar{t}}^{t} \big(\langle -A(s)x + G(\sigma, \pi_1(u(\sigma)), \pi_2(u(\sigma)) - x\rangle_+ \\
&\quad + \alpha\|\pi_2(u(\sigma)) - x\| + C\|h(\sigma) - h(s)\|\big)d\sigma.
\end{aligned}$$

Since $u(t)$ is a translation, $\pi_1(u(t)) = x_t$, where

$$x(t) = \begin{cases} \phi(t), & -r \le t < 0, \\ \pi_2(u(t)), & 0 \le t \le T. \end{cases}$$

Therefore,

$$\begin{aligned} &\|x(t) - x\| - \|x(\bar{t}) - x\| \\ &\le \int_{\bar{t}}^{t} \big(\langle -A(s)x + G(\sigma, x_\sigma), x(\sigma) - x \rangle_+ \\ &\qquad + \alpha\|x(\sigma) - x\| + C\|h(\sigma) - h(s)\|\big) d\sigma. \end{aligned}$$

It implies that $x(t)$ is an integral solution of (FDE).

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