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ULTRASEPARABILITY OF CERTAIN FUNCTION ALGEBRAS

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Throughout this paper, let X be a compact Hausdorff space, and let $C(X)$ (resp. $C_{\mathbf{R}}(X)$) be the complex (resp. real) Banach algebra of all continuous complex-valued (resp. real-valued) functions on X with the pointwise operations and the supremum norm $\| \ \|_X$. A *Banach function algebra on* X is a Banach algebra lying in $C(X)$ which separates the points of X and contains the constants. A Banach function algebra on X equipped with the supremum norm is called a *uniform algebra on* X, that is, a uniformly closed subalgebra of $C(X)$ which separates the points of X and contains the constants.

Let E be a (real or complex) normed linear space with norm $\| \ \|_E$. Denote by $\tilde{E} = \ell_\infty(\mathbf{N}, E)$ the space of all bounded functions from the set $\mathbf{N} = \{1, 2, 3, \dots\}$ to E normed as follows:

$$\|\tilde{f}\|_{\tilde{E}} \equiv \sup\{\|f_n\|_E : n \in \mathbf{N}\} < \infty$$

for a sequence $\tilde{f} = \{f_n\}_{n=1}^{\infty}$ in $\tilde{E}$.

Denote by $\tilde{X} = \beta(\mathbf{N} \times X)$ the Stone-Čech compactification of the product space $\mathbf{N} \times X$. Since every sequence $\{f_n\}_{n=1}^{\infty}$ in $\ell_\infty(\mathbf{N}, C(X))$ can be considered as a function from $\mathbf{N} \times X$ to $\mathbf{C}$, it has a unique continuous extension to a function in $C(\tilde{X})$. So, we have $\ell_\infty(\mathbf{N}, C(X)) = C(\tilde{X})$.

Let E be a (real or complex) normed linear space continuously injected in $C(X)$. We say that E is *ultraseparating on* X if $\tilde{E}$ separates the points of $\tilde{X}$.

Let A be a Banach function algebra on X and let F be a nonempty closed subset of X. Then $A|_F$ with the quotient norm

$$\|f\|_{A|_F} \equiv \inf\{\|g\|_A : g \in A, \ g|_F = f\}$$

is a Banach function algebra on F.

The following lemma is a part of Lemma 6 of O. Hatori [3].

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LEMMA. *Let A be a uniform algebra on X. Then the following are equivalent:*

(1) *A is ultraseparating on X.*
(2) *For every $x \in X$, there exists a compact neighborhood F of x such that $A|_F$ is ultraseparating on F with respect to the quotient norm.*

Let A be a non-empty subset of $C(X)$. A set K in X is an *antisymmetric* set for A if K is non-empty and every f in A which is real-valued on K is constant on K. A is *antisymmetric* provided that X is an antisymmetric set for A.

Let A be a uniform algebra on X. Bishop's Antisymmetric Decomposition Theorem says that X is the union of maximal antisymmetric sets $\{K_\alpha\}$ for A which are closed and mutually disjoint. Moreover, for each K_α, the restriction algebra $A|_{K_\alpha}$ is a uniform algebra on K_α.

PROPOSITION. *Let A be a uniform algebra on X. Suppose that for every maximal antisymmetric set K for A, $A|_K$ is ultraseparating on K with respect to the quotient norm. Then A is ultraseparating on X.*

Proof. Let $x \in X$ be given. Then $x \in K$ for some maximal antisymmetric set K for A. By the hypothesis, $A|_K$ is ultraseparating on K, so by the above lemma of Hatori, there exists a compact neighborhood F of x in K such that $(A|_K)|_F$ is ultraseparating. Since $(A|_K)|_F = A|_F$ and $\| \ \|_{(A|_K)|_F} = \| \ \|_{A|_F}$, x has a compact neighborhood F such that $A|_F$ is ultraseparating. By the lemma again, A is ultraseparating on X.

In [1], A. Bernard proved the following fact (More generally, see Lemma 4.10 of [2].):

(*) If a uniform algebra A on X is ultraseparating, then for every proper closed subset Y of X, $A|_Y$ is ultraseparating on Y.

The following corollary is the converse of (*), which generalizes Corollary 2.4 of [4].

COROLLARY 1. *Let A be a uniform algebra on X. If for every proper closed subset Y of X $A|_Y$ is ultraseparating on Y, then A is ultraseparating on X.*

It is not known whether the uniform closure $A\bar{\otimes}B$ of the tensor product $A \otimes B$ of ultraseparating uniform algebras A and B on X and Y, respectively, is ultraseparating or not. In [5], S. Sidney showed the answer is *yes* if $B = C(Y)$ where Y is totally disconnected.

The following corollary shows that total disconnectedness of Y is unnecessary for $B = C(Y)$ if A is not antisymmetric.

COROLLARY 2. *Let X and Y be compact Hausdorff spaces, and let A be a non-antisymmetric ultraseparating uniform algebra on X. Then $A\bar{\otimes}C(Y)$ is ultraseparating on $X \times Y$.*

Proof. First, note that every maximal antisymmetric set for $A\bar{\otimes}C(Y)$ is of the form $K \times \{y\}$ for some maximal antisymmetric set K for A and for some $y \in Y$. So, by the Proposition it suffices to show that $A\bar{\otimes}C(Y)|_{K\times\{y\}}$ is ultraseparating for all maximal antisymmetric set K for A and for all $y \in Y$. But since $A\bar{\otimes}C(Y)|_{K\times\{y\}} = A|_K$, by $(*)$ $A|_K$ is ultraseparating on K, so we are done.

It is well-known for a uniform algebra A on X that there exists a unique minimal closed subset E of X with the property that A contains every continuous function on X which vanishes on E. This set E is call the *essential* set for A.

COROLLARY 3. *Let A be a uniform algebra on X. If A is ultraseparating on its essential set, then A is ultraseparating on X.*

Proof. Let K be a maximal antisymmetric set for A. If K is a singleton,then $A|_K = C(K)$ is ultraseparating on K. If K is not a singleton, then K is contained in the essential set for A, hence $A|_K$ is ultraseparating on K. Thus, A is ultraseparating on X by the Proposition.

EXAMPLE. (due to O. Hatori) Let

$$D = \{\ z \in \mathbf{C}\ :\ |z| = 1\ \},$$

and let $A(D)$ is the disk algebra, that is, the algebra of all continuous complex-valued functions on D which have continuous extensions to the closed unit disk which are analytic on the open unit disk.

Put $X = \{ z \in \mathbf{C} \ : \ 1 \leq |z| \leq 2 \}$, and define

$$A = \{ f \in C(X) \ : \ f|_D \in A(D) \}.$$

Then A is a uniform algebra on X whose essential set is D. Since $A|_D$ is ultraseparating on D, A is ultraseparating on X by Corollary 3.

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