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A NOTE ON APPROXIMATION PROPERTIES OF BANACH SPACES

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1. Introduction

It is well known that the approximation property and the compact approximation property are not hereditary properties; that is, a closed subspace M of a Banach space X with the (compact) approximation property need not have the (compact) approximation property. In 1973, A. Davie [2] proved that for each $2 < p < \infty$, there is a closed subspace Y_p of ℓ_p which does not have the approximation property. In fact, the space Davie constructed even fails to have a weaker property, the compact approximation property. In 1991, A. Lima [12] proved that if X is a Banach space with the approximation property and a closed subspace M of X is locally λ-complemented in X for some $1 \leq \lambda < \infty$, then M has the approximation property.

In Section 3, we will see that the compact approximation property analogue of Lima's result holds for reflexive Banach spaces. In Theorem 5 we will prove that if X is a reflexive Banach space with the compact approximation property and M is locally λ-complemented in X for some $1 \leq \lambda < \infty$, then M has the compact approximation property.

A. Grothendieck [7] proved that if X is a reflexive Banach space or a separable conjugate space which has the approximation property, then X has the metric approximation property. In the case of the compact approximation property, we have the following analogue ; if X is a separable reflexive Banach space which has the compact approximation property, then X has the metric compact approximation property [1]. However, in general the approximation property or the bounded approximation property does not imply the metric approximation property [6].

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In Theorem 4 we will prove that if a Banach space X has the approximation property and the metric compact approximation property, then X has the metric approximation property.

2. Preliminaries

If X and Y are Banach spaces, $L(X,Y)$ (respectively, $K(X,Y)$) will denote the space of all bounded linear operators (respectively, compact linear operators) from X to Y. If $X = Y$, then we will simply write $L(X)$ (respectively, $K(X)$). If X is a Banach space, B_X will denote the closed unit ball of X.

A Banach space X is said to have the *approximation property* (respectively, *compact approximation property*) if for every compact set K in X and every $\varepsilon > 0$, there exists a finite rank operator (respectively, compact linear operators) $T : X \to X$ such that $\|Tx - x\| < \varepsilon$ for all $x \in K$. For $1 \leq \lambda < \infty$ a Banach space X is said to have the *λ-approximation property* if for every compact set K in X and every $\varepsilon > 0$ there is a finite rank operator T in X such that $\|T\| \leq \lambda$ and $\|Tx - x\| < \varepsilon$ for all $x \in K$. A Banach space X is said to have the *bounded approximation property* if it has the λ-approximation property for some $1 \leq \lambda < \infty$. A Banach space X is said to have the *metric approximation property* if it has the 1-approximation property. The metric compact approximation property is defined in an obvious way.

If X and Y are Banach spaces, the space $L(X,Y)$ endowed with the topology τ of uniform convergence on compact sets in X is a locally convex space generated by seminorms of the form $\|T\|_K = \sup\{\|Tx\| : x \in K\}$, where K ranges over all compact subsets of X. In what follows the topology τ on $L(X,Y)$ will always denote the above topology.

Continuous linear functionals on $(L(X,Y),\tau)$ have concrete representations.

PROPOSITION 1 [13]. *Let X and Y be Banach spaces. The continuous linear functionals on $(L(X,Y),\tau)$ consists of all functionals of the form*

$$\Phi(T) = \sum_{i=1}^{\infty} y_i^*(Tx_i),$$

where $x_i \in X$, $y_i^* \in Y^*$ and $\sum_{i=1}^{\infty} \|x_i\| \, \|y_i^*\| < \infty$.

A closed subspace M of a Banach space X is said to have *locally λ-complemented* $(1 \leq \lambda < \infty)$ if for any $\varepsilon > 0$ and a finite dimensional subspace F of X there exists an operator $T : F \to M$ such that

(i) $Tx = x$ for all $x \in F \cap M$
(ii) $\|T\| \leq \lambda + \varepsilon$

PROPOSITION 2 [12]. *If M is a closed subspace of a Banach space X, then M is locally λ-complemented if and only if there exists a norm-λ projection in X^* with kernel $M^\perp$, the annihilator of M in X^*.*

3. Results

It is clear that a Banach space X has the compact approximation property (respectively, the metric compact approximation property) if and only if the identity operator on X is in the τ-closure of $K(X)$ (respectively, $B_{K(X)}$).

The following characterization of the metric compact approximation is an analogue of a characterization of the metric approximation property [13, P.39]. The proof given below actually is the Lindenstrauss-Tzafriri proof of a characterization of the approximation property [13, P.32].

THEOREM 3. *For a Banach space X, the following assertions are equivalent.*

(i) *X has the metric compact approximation property.*
(ii) *$B_{K(X)}$ is τ-dense in $B_{L(X)}$.*
(iii) *For every choice of $\{x_n\} \subseteq X$ and $\{x_n^*\} \subseteq X^*$ such that $\sum_{n=1}^{\infty} \|x_n\| \, \|x_n^*\| < \infty$ and $|\sum_{n=1}^{\infty} x_n^*(Tx_n)| \leq 1$ for all T in $B_{K(X)}$ we have $|\sum_{n=1}^{\infty} x_n^*(x_n)| \leq 1$.*
(iv) *$B_{K(X,Y)}$ is τ-dense in $B_{L(X,Y)}$ for every Banach space Y.*
(v) $B_{K(Y,X)}$ is τ-dense in $B_{L(Y,X)}$ for every Banach space Y.

Proof. The implication (iv) or (v) $\Rightarrow$ (ii) is obvious. For (ii) $\Rightarrow$ (iv) let $T \in B_{L(X,Y)}$, $\varepsilon > 0$ and K be a compact subset of X. Since $B_{K(X)}$ is τ-dense in $B_{L(X)}$, there exists S in $B_{K(X)}$ such that

$$\|Sx - x\| < \varepsilon$$

for every $x \in K$. Since $\|TSx - Tx\| < \epsilon$ for every $x \in K$ and TS is in $B_{K(X,Y)}$, $B_{K(X,Y)}$ is τ-dense in $B_{L(X,Y)}$. To prove (ii) $\Rightarrow$ (v) let $T \in B_{L(Y,X)}$, $\varepsilon > 0$ and K be a compact subset of Y. Since $B_{K(X)}$ is τ -dense in $B_{L(X)}$ and $T(K)$ is a compact subset of X, there exists $S \in B_{K(X)}$ such that

$$\|Sy - y\| < \varepsilon$$

for all $y \in T(K)$. Since $\|STx - Tx\| < \varepsilon$ for all $x \in K$ and $ST \in B_{K(Y,X)}$, $B_{K(Y,X)}$ is τ-dense in $B_{L(Y,X)}$. This proves that (ii) $\Rightarrow$ (v). Suppose (i) holds. Let $T \in B_{L(X)}$ and K be a compact subset of X. Since $T(K)$ is a compact subset of X, for any $\varepsilon > 0$ there exists $T_1 \in B_{K(X)}$ such that

$$\|T_1 Tx - Tx\| < \varepsilon$$

for all $x \in K$. Since $T_1 T \in B_{K(X)}$, this proves the implication (i) $\Rightarrow$ (ii). (iii) follows from (ii) and Proposition 1. So it remains to prove that (iii) implies (i).

Suppose X does not have the metric compact approximation property. Then the identity map I on X is not in the τ-closure $\overline{B_{K(X)}}^{\tau}$ of $B_{K(X)}$. Since $\overline{B_{K(X)}}^{\tau}$ is a closed convex and balanced subset in the locally convex space $(L(X), \tau)$ and does not contain the identity map I on X, by the separation theorem there is a positive number α and a continuous linear functional Φ on $(L(X), \tau)$ such that

$$|Re\Phi(T)| \leq \alpha < Re\Phi(I)$$

for all T in $B_{K(X)}$. Replacing Φ by $\frac{1}{\alpha}\Phi$, we may assume that $\alpha = 1$ in the above inequality. Since $B_{K(X)}$ is balanced, we have

$$|\Phi(T)| \leq 1 < |\Phi(I)|$$

for all T in $B_{K(X)}$. Since Φ is a τ-continuous linear functional on $L(X)$, by Proposition 1 there exist sequences $\{x_n\} \subseteq X$ and $\{x_n^*\} \subseteq X^*$ such that $\sum_{n=1}^{\infty} \|x_n\| \, \|x_n^*\| < \infty$ and $\Phi(T) = \sum_{n=1}^{\infty} x_n^*(Tx_n)$ for every T in $L(X)$. Thus we have

$$|\sum_{n=1}^{\infty} x_n^*(Tx_n)| \leq 1 < |\sum_{n=1}^{\infty} x_n^*(x_n)|$$

for all $T \in B_{K(X)}$. Thus (iii) does not hold and the proof of (iii) $\Rightarrow$ (i) is complete.

We are now ready to prove the main results of this paper.

The following theorem relates the approximation property and the metric approximation property.

THEOREM 4. *If X is a Banach space which has the approximation property and also the metric compact approximation property, then X has the metric approximation property.*

Proof. Let $F(X)$ denote the space of finite rank operators on X. If $T \in K(X)$, then $T(B_X)$ has the compact closure. Since X has the approximation property, given $\varepsilon > 0$ there exists a finite rank operator T_1 such that

$$\|T_1 y - y\| < \varepsilon$$

for all $y \in T(B_X)$. Thus $\|T_1 T - T\| < \varepsilon$. Since $T_1 T$ is in $F(X)$, it follows that $F(X)$ is dense in $K(X)$ in the operator norm topology. Thus $B_{F(X)}$ is dense in $B_{K(X)}$ in the operator norm topology. Since the operator norm topology on $L(X)$ is larger than the topology τ of uniform convergence on compact sets we have $\overline{B_{F(X)}}^{\tau} = \overline{B_{K(X)}}^{\tau}$. Since X has the metric compact approximation property, by Theorem 3 $\overline{B_{K(X)}}^{\tau} = B_{L(X)}$. Hence, $\overline{B_{F(X)}}^{\tau} = B_{L(X)}$ and X has the metric approximation property.

THEOREM 5. *Suppose X is a reflexive Banach space with the compact approximation property. If a closed subspace M of X is locally λ-complemented in X, then M has the compact approximation property. Moreover, if X has the δ-compact approximation property, then M has the $\delta\lambda$-compact approximation property.*

Proof. By Proposition 2, there exists a projection P on X^* with kernel $M^{\perp}$ and $\|P\| = \lambda$. Then $P^* : X^{**} \to X^{**}$ has the range $M^{\perp\perp}$. Since X is reflexive, $M^{\perp\perp} = M$ and $Q = P^*$ is a projection on X with the range M and $\|Q\| = \lambda$.

Let K be a compact subset of M and $\varepsilon > 0$. Since X has the compact approximation property, there exists a compact operator $T : X \to X$ such that

$$\|Tx - x\| < \frac{\varepsilon}{\lambda}$$

for all $x \in K \subseteq M$. Since the projection Q has the range M, we get that

$$\|QTx - x\| < \varepsilon$$

for all $x \in K$ and QT restricted to M is in $K(M)$. Thus M has the compact approximation property.

If $\|T\| \leq \delta$, then $\|QT\| \leq \delta\lambda$ and the proof is complete.

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