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GLOBALLY DETERMINED ALGEBRAS

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0. Introduction

This paper is a contribution to the study of the isomorphism problems for algebras. Among the isomorphism problems, that of global determination is investigated here. That is, our investigation of the problems is concerned with the question whether two algebras are isomorphic when their globals are isomorphic. The answer is not always affirmative. The counterexample, due to E. M. Mogiljanskaja, is the class of all infinite semigroups. But T. Tamura and J. Shafer [6] proved that the class of all groups is globally determined and announced the same result for the class of rectangular bands. Važenin [7] proved that for any set X, the transformation semigroup T_X must be isomorphic to any semigroup S for any $\mathcal{P}(S) \simeq \mathcal{P}(T_X)$.

We will show that the class of all Heyting algebras is globally determined, directly. Moreover we will investigate to the class of all bounded lattices and the class of all generalized Boolean algebras.

The rest of this paper is divided into two sections. In section 1, we will give some basic definitions and facts. And finally in the last section we shall prove that the referred class are globally determined. For standard concepts and facts from lattice theory, we refer the reader to Grätzer[4]. However we use $+$ and $\cdot$ instead of $\vee$ and $\wedge$ for the lattice operation.

1. Preliminaries

Given an algebra $\mathcal{A} = (A, \mathcal{F})$ where $\mathcal{F}$ is a set of operations, we define the *global*, $\mathcal{P}(A)$, of the $\mathcal{A}$ to be the family of all complexes of A with operation given by

$$f(B_1, \dots, B_n) = \{f(b_1, \dots, b_n) : \ b_i \in B_i\}$$

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whenever f is an n-ary operation belonging to $\mathcal{F}$ and each B_i is a complex of A $(i = 1, 2, \ldots, n)$.

We call a class $\mathcal{K}$ of algebra *globally determined* if any two members of $\mathcal{K}$ having isomorphic globals must themselves be isomorphic.

We now consider some basic definitions and notations, and state several basic properties.

DEFINITION 1.1. An algebra $\langle H, +, \cdot, \rightarrow, 0, 1\rangle$ with three binary and two nullary operations is a Heyting algebra if it satisfies:

(i) $\langle H, +, \cdot\rangle$ is a distributive lattice.
(ii) $x \cdot 0 = 0$ and $x + 1 = 1$.
(iii) $x \rightarrow x = 1$.
(iv) $(x \rightarrow y)\cdot y = y$ and $x\cdot(x \rightarrow y) = x\cdot y$. $x \rightarrow y\cdot z = (x \rightarrow y)\cdot(x \rightarrow z)$ and $(x + y) \rightarrow z = (x \rightarrow z) \cdot (y \rightarrow z)$ for any x, y and $z \in H$.

DEFINITION 1.2. A generalized Boolean algebra is a relatively complemented distributive lattice with a bottom element 0.

Let L be a lattice and let $I(L)$ denote the set of all ideals of L and let $I_0(L) = I(L) \cup \phi$.

LEMMA 1.3. *$I(L)$ and $I_0(L)$ are posets under inclusion and as posets they are lattices.*

In fact $I \bigoplus J = (I \cup J]$ if we agree that $(\phi] = \phi$ where $(H]$ denote the ideal generated by a subset H and $\bigoplus$ denotes the join operation in $I(L)$. And $I \cdot J = I \cap J$ for $I, J \in I(L)$.

In general,

$$\bigoplus(I_\lambda \mid \lambda \in \Lambda) = (\bigcup(I_\lambda \mid \lambda \in \Lambda)]$$

where $\bigoplus I_\lambda$ denote the join of all $I'_\lambda s$.

COROLLARY 1.4. *Let $I_\lambda, \lambda \in \Lambda$ be ideal and let $I = \bigoplus(I_\lambda \mid \lambda \in \Lambda)$. Then $i \in I$ if and only if $i \leq j_{\lambda_0} + j_{\lambda_1} + \cdots + j_{\lambda_{n-1}}$ for some integer $n \geq 1$ and for some $\lambda_0, \ldots, \lambda_{n-1} \in \Lambda$, $j_{\lambda_i} \in I_{\lambda_i}$ $(i = 0, \ldots, n-1)$.*

Now we observe that the formula:

$$(a] \oplus (b] = (a + b], \qquad (a] \odot (b] = (a \cdot b]$$

where $\odot$ is the meet operation in $I(L)$.

Since $a \neq b$ implies that $(a] \neq (b]$, these formulas yield following corollary:

COROLLARY 1.5. *A lattice L can be embedded in $I(L)$ and $a \to (a]$ is an embedding.*

From above corollary, we note that a lattice L is isomorphic to the set of all principal ideals in L.

LEMMA 1.6. *Let I be an ideal in a lattice L. Then I is principal if and only if, for any updirected family $\{I_k : k \in \Lambda\}$ of ideals, $I \leq \bigoplus_{k\in\Lambda} I_k$ implies that $I \leq I_j$ for $j \in \Lambda$.*

For any algebra $\mathcal{A} = (A, \mathcal{F})$, a singleton member of $\mathcal{P}(\mathcal{A})$ will frequently be identified with the element it contains.

2. Main Results

LEMMA 2.1. *Every isomorphism between globals of lattice which have a top and a bottom element maps a top and a bottom element to a top a bottom element, respectively.*

Proof. Let L_1 and L_2 be lattices with a top and a bottom element, 1, 0, and $1'$, $0'$, respectively. Let $\Psi : \mathcal{P}(L_1) \longrightarrow \mathcal{P}(L_2)$ be an isomorphism. Then for any $Y \in \mathcal{P}(L_2)$, there exists $X \in \mathcal{P}(L_1)$ such that $\Psi(X) = Y$. Thus we have $\Psi(1) + Y = \Psi(1) + \Psi(X) = \Psi(1 + X) = \Psi(1)$ and $\Psi(1) + 1' = 1'$. Hence $\Psi(1) = 1'$. Also we have $\Psi(0) \cdot Y = \Psi(0) \cdot \Psi(X) = \Psi(0 \cdot X) = \Psi(0)$ and $\Psi(0) \cdot 0' = 0'$. Thus $\Psi(0) = 0'$. The proof is complete.

THEOREM 2.2. *Let $\langle H_1, +, \cdot, \to, 0, 1\rangle$ and $\langle H_2, +, \cdot, \to, 0', 1'\rangle$ be Heyting algebras. If $\mathcal{P}(H_1) \cong \mathcal{P}(H_2)$, then $H_1 \cong H_2$. Moreover every isomorphism of $\mathcal{P}(H_1)$ and $\mathcal{P}(H_2)$ restricts to an isomorphism of H_1 and H_2.*

Proof. Let $\Psi : \mathcal{P}(H_1) \longrightarrow \mathcal{P}(H_2)$ be an isomorphism. Let $x \in H_1$ and $A = \Psi(x)$. Then we show that A is a singleton. Since $x \to x = 1$, by lemma 2.1, we have $A \to A = 1'$. Thus for any $a, b \in A$, $a \to b = 1' = b \to a$. So we have $a = a \cdot 1' = a \cdot (a \to b) = a \cdot b = b \cdot a = b \cdot (b \to a) = b \cdot 1' = b$, and hence A is a singleton. Therefore Ψ maps singletons to singletons. Similarly, Ψ^{-1} maps singletons to singletons. The proof is complete.

LEMMA 2.3. *Let L_1 and L_2 be lattices having a top element 1 and $1'$, respectively. And let $\Psi : \mathcal{P}(L_1) \longrightarrow \mathcal{P}(L_2)$ be an isomorphism. Then $\Psi(L_1) \cup \{1'\}$ is a filter of L_2.*

Proof. Let K be a nonempty subset of L_1 where $\Psi(K) = \Psi(L_1) \cup \{1'\}$. Since Ψ is an isomorphism, $\Psi(L_1) \cup \{1'\}$ is a sublattice of L_2. Thus K is a sublattice of L_1. Since $\Psi(K) = \Psi(K + L_1)$, $K = K + L_1$. Therefore K is a filter of L_1. Let $T \in \mathcal{P}(L_1)$ where $\Psi(T) = L_2$. Since K is a filter of L_1, $K + T \subset K$. Also since $1' \in \Psi(K)$, $\Psi(K) \cdot L_2 = L_2$. Hence $\Psi(K \cdot T) = \Psi(T)$. Thus $K \cdot T = T$. Let $k \in K$. Then $k \in K + T$ because $k = k + k \cdot t$ for any $t \in T$. Thus $K \subset K + T$. so we have $K = K + T$. Thus $\Psi(K) = \Psi(K+T) = \Psi(K) + \Psi(T) = \Psi(K) + L_2$. Therefore $\Psi(K)$ is a filter of L_2.

Dually, we have the following corollary.

COROLLARY 2.4. *Let L_1 and L_2 be lattices having a bottom element 0 and $0'$, respectively. And let $\Psi : \mathcal{P}(L_1) \longrightarrow \mathcal{P}(L_2)$ be an isomorphism. Then $\Psi(L_1) \cup \{0'\}$ is an ideal of L_2.*

LEMMA 2.5. *Let L_1 and L_2 be bounded lattices and let Ψ be an isomorphism from $\mathcal{P}(L_1)$ onto $\mathcal{P}(L_2)$. Then $\Psi(L_1) \cup \{0', 1'\} = L_2$ where $0'$ and $1'$ are a bottom element and a top element of L_2, respectively.*

Proof. If L_2 is a chain of two elements, then, by lemma 2.1, $\Psi(L_1) = L_2$. We are done. Assume there exists an element $x (\neq 0', 1')$ of L_2. Suppose $x + y = 1'$ for any $y \in \Psi(L_1)$. Let $A \in \mathcal{P}(L_1)$ with $\Psi(A) = \{x\}$. Then $\Psi(A) + \Psi(L_1) = \{1'\}$. By lemma 2.1, $A + L_1 = \{1\}$ where 1 is a top element of L_1. Thus $A = \{1\}$. Therefore $x = 1'$. We have a contradiction. Thus there exists an element y of $\Psi(L_1)$ such that $x + y \neq 1'$. Let $z = x + y$. Then $z \in \Psi(L_1) \cup \{1'\}$ because, by lemma 2.3, $\Psi(L_1) \cup \{1'\}$ is a filter of L_2. Since $z \neq 1'$, $z \in \Psi(L_1)$. By corollary 2.4, $\Psi(L_1) \cup \{0'\}$ is an ideal of L_2. Hence $x \in \Psi(L_1) \cup \{0'\}$. Therefore $\Psi(L_1) \cup \{0', 1'\} = L_2$. The proof is complete.

LEMMA 2.6. *Let L_1 and L_2 be bounded lattices and let Ψ be an isomorphism form $\mathcal{P}(L_1)$ onto $\mathcal{P}(L_2)$. If I is a proper ideal of L_1, then $\Psi(I)$ is an ideal of L_2.*

Proof. Let $T, W, K \in \mathcal{P}(L_1)$ with $\Psi(T) = L_2$, $\Psi(W) = \Psi(L_1) \cup \{0'\}$ and $\Psi(K) = \Psi(L_1) \cup \{1'\}$ where $0'$ and $1'$ are a bottom element and a top element of L_2. Then T, W, and K are sublattices of L_1 because Ψ is an isomorphism. Since $\Psi(L_1) \cdot (\Psi(L_1) \cup \{0'\}) = \Psi(L_1) \cup \{0'\}$, $L_1 \cdot W = W$. Thus W is an ideal of L_1. Also $L_1 + K = K$ because $\Psi(L_1) + (\Psi(L_1) \cup \{1'\}) = \Psi(L_1) \cup \{1'\}$. Thus K is a filter of L_1.

Claim 1: $K \cap W = T$.

By lemma 2.5, $L_2 = \Psi(L_1) \cup \{0', 1'\}$. Hence $L_1 \cdot T = W$. Thus $T \subset W$. Also $L_1 + T = K$. Thus $T \subset K$. So we get $T \subset K \cap W$. Now show that $K \cap W \subset T$. Let $x \in K \cap W$. Then $x \in K$ and $x \in W$. Since $T \cdot K = T$ and $T + W = T$, $x + x \cdot t \in T$ for any $t \in T$. Hence $x \in T$. Thus $K \cap W \subset T$. Therefore $K \cap W = T$.

Claim 2: $K \cup W = L_1$.

Let $P = K \cup W$. Since $K + T = K$ and $W + T = T$, $P + T = K$. Hence $\Psi(K) = \Psi(P+T) = \Psi(P) + \Psi(T) = \Psi(P) + (\Psi(L_1) \cup \{0', 1'\}) = \Psi(L_1) \cup \{1'\} \cup \Psi(P) = \Psi(K) \cup \Psi(P)$. Thus $\Psi(P) \subset \Psi(K)$. Since $P \cdot T = W$, $\Psi(W) = \Psi(P) \cdot (\Psi(L_1) \cup \{0', 1'\}) = \Psi(L_1) \cup \Psi(P) \cup \{0'\} = \Psi(W) \cup \Psi(P)$. Thus $\Psi(P) \subset \Psi(W)$. Also since $P + W = P$, $\Psi(P) = \Psi(P + W) = \Psi(P) + \Psi(W) = \Psi(L_1) \cup \Psi(P)$. So $\Psi(L_1)) \subset \Psi(P)$. Thus $\Psi(L_1) \subset \Psi(P) \subset \Psi(K) \cap \Psi(W) = \Psi(L_1)$. Therefore $K \cup W = L_1$.

Claim 3: $T \cup \{0, 1\} = L_1$ where 0 and 1 are a bottom element and a top element of L_1.

Let $A = T \cup \{0\}$. Then $A + W = W$, $A \cdot T = A$, $A + L_1 = L_1$, and $A \cdot L_1 = W$. Hence $\Psi(W) = \Psi(A) + \Psi(W) = \Psi(A) + (\Psi(L_1) \cup \{0'\}) = \Psi(L_1) \cup \Psi(A)$. So $\Psi(A) = \Psi(W)$. Therefore $A = W$. Let $B = T \cup \{1\}$. Then $B + W = B$, $B \cdot K = K$, $B + L_1 = K$, and $B \cdot L_1 = L_1$. Hence $\Psi(B) = \Psi(B) + \Psi(W) = \Psi(B) + (\Psi(L_1) \cup \{0'\}) = \Psi(K) \cup \Psi(B)$. Thus $\Psi(K) \subset \Psi(B)$. Also $\Psi(K) = \Psi(B) \cdot \Psi(K) = \Psi(B) \cdot (\Psi(L_1) \cup \{(1'\}) = \Psi(L_1) \cup \Psi(B)$. Thus $\Psi(B) \subset \Psi(K)$. So $\Psi(B) = \Psi(K)$. Thus $B = K$. Therefore $A \cup B = K \cup W$. By claim 2, $A \cup B = L_1$. Also $A \cup B = T \cup \{0\} \cup T \cup \{1\} = T \cup \{0, 1\}$. Therefore $T \cup \{0, 1\} = L_1$.

Let I be a proper ideal of L_1. Clearly $\Psi(I)$ is a sublattice of L_2. So it suffices to show that $\Psi(I) \cdot L_2 = \Psi(I)$. By claim 3, $I \subset T \cup \{0\}$. Hence $I \subset I \cdot T$. Also $I \cdot T \subset I$ because I is an ideal of L_1. Thus $I \cdot T = I$. Hence $\Psi(I) = \Psi(I \cdot T) = \Psi(I) \cdot \Psi(T) = \Psi(I) \cdot L_2$. Therefore $\Psi(I)$ is an ideal of L_2. The proof is complete.

THEOREM 2.7. *Let L_1 and L_2 be bounded lattices. If $\mathcal{P}(L_1) \cong \mathcal{P}(L_2)$, then $L_1 \cong L_2$.*

Proof. Let $\Psi : \mathcal{P}(L_1) \longrightarrow \mathcal{P}(L_2)$ be an isomorphism. By Corollary 1.5, a lattice is isomorphic to the lattice of its principal ideals. $I \cap J = I \cdot J$ and, by lemma 1.6, the principal ideals of L are the compact elements of the ideal lattice $I(L)$. Also by lemma 2.1, $\Psi(\{1\}) = \{1'\}$. Thus it is enough to show that Ψ and Ψ^{-1} map proper ideals to proper ideals. Since Ψ is an isomorphism, we may deal only with Ψ. Let I and J be proper ideals in L_1 such that $I \subseteq J$. Then by lemma 2.6 $\Psi(I)$ and $\Psi(J)$ are ideals in L_2. Also $\Psi(I) \subseteq \Psi(J)$. Thus we have two cases.

a) $\Psi(L_1) = L_2$.

b) $\Psi(L_1)$ is not an ideal in L_2.

Thus $\Psi(I)$ and $\Psi(J)$ are proper ideals in L_2. The proof is complete.

LEMMA 2.8. *Let B_1 and B_2 be generalized Boolean algebras and $\Psi : \mathcal{P}(B_1) \to \mathcal{P}(B_2)$ be an isomorphism. Then $0 \in \Psi(B_1)$ where 0 is a bottom element of B_2.*

Proof. If B_2 is a chain of two elements or one element, then by lemma 2.1, $\Psi(B_1) = B_2$. So we are done. Now we will prove this lemma in $B_2 > 2$

Claim that $\Psi(B_1)$ contains at least two comparable non-zero elements. If $\Psi(B_1) = y_1$ for some $y_1 \in B_2$, then y_1 is non-zero because $\Psi(0_{L_1}) = 0$ by Lemma 2.1. Since, by corollary 2.4, $\Psi(B_1) \cup 0$ is an ideal of B_2, y_1 is an atom of B_2. Also since B_2 has more than two elements, there exists $y_2 \in B_2$ such that $y_2 \neq y_1$ and $y_1 \leq y_2$. Since B_2 is a generalized Boolean algebra, there exists $y_1' \in B_2$ such that $y_1 \cdot y_1' = 0$ and $y_1 + y_1' = y_2$ and we note that y_1' is non-zero. Let $\Psi(K) = y_1'$. Then $\Psi(B_1) \cdot \Psi(K) = 0$. Thus $B_1 \cdot K = 0_{L_1}$ and hence $K = 0_{L_1}$. Thus We get a contradiction. If $\Psi(B_1) = \{0, y_1\}$ for some $y_1 \in B_2$, then by the sane method above, we get a contradiction. Thus $\Psi(B_1)$ contain at least two comparable non-zero elements. By claim, there exist $x, y \in \Psi(B_1)$ such that x, y are non-zero elements and $x \leq y$. Now $x \in [0, y]$. Thus there exists $x'(\neq 0) \in B_2$ such that $x \cdot x' = 0$, $x + x' = y$. Since $\{0\} \cup \Psi(B_1)$ is an ideal, $x' \leq y$ and $\Psi(B_1)$ is a sublattice of B_2, then $0 = x \cdot x' \in \Psi(B_1)$. The proof is complete.

THEOREM 2.9. *Let B_1 and B_2 be generalized Boolean algebras. If $\mathcal{P}(B_1) \simeq \mathcal{P}(B_2)$, then $B_1 \simeq B_2$. Moreover, every isomorphism of $\mathcal{P}(B_1)$ onto $\mathcal{P}(B_2)$ restricts to an isomorphism of B_1 onto B_2.*

Proof. Let $\Psi : \mathcal{P}(B_1) \to \mathcal{P}(B_2)$ be an isomorphism. As referred in the proof of theorem 2.7, it is enough to show that Ψ map ideals to ideals. Since I is an ideal in B_1, $I+I = I$. Thus $\Psi(I)$ is a sublattice of B_2. Next we want to show that for any $a \in \Psi(I)$ and $b \in B_2$, $a \cdot b \in \Psi(I)$. It suffices to show that $\Psi(I) \cdot B_2 = \Psi(I)$. Let $\Psi(A) = B_2$. Since I is an ideal in B_1, $I \cdot A \subseteq I$. Since I is an ideal in B_1, $\Psi(I) \cdot \Psi(B_1) = \Psi(I \cdot B_1) = \Psi(I)$. Also, by Lemma 2.8, $0 \in \Psi(B_1)$. Hence $0 \in \Psi(I)$. Thus $B_2 \subseteq \Psi(I) + B_2$. Obviously $\Psi(I) + B_2 \subseteq B_2$. Thus $\Psi(I) + B_2 = B_2$ and hence $I + A = A$. Now for any $i \in I$, $i + a \in A$. So $I \subseteq I \cdot A$ because $i = i(i + a)$. Thus $\Psi(I) \cdot B_2 = \Psi(I)$. Therefore $\Psi(I)$ is an ideal in B_2. The proof is complete.

COROLLARY 2.10. *The class of all Boolean algebras is globally determined.*

Proof. By the fact that a Boolean algebra is a generalized Boolean algebra having a top element.

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