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DECOMPOSITIONS OF IDEALS IN BCI-ALGEBRAS

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In 1966, Iséki [4] introduced the notion of BCI-algebras which is a generalization of BCK-algebras. The ideal theory plays an important role in studying BCK/BCI-algebras.

In this paper we study decompositions of ideals in BCI-algebras, and give a characterization of closed ideals. Also we define ignorable ideals in BCI-algebras, and investigates its properties.

Let us review some definitions and results.

By a BCI-algebra we mean a nonempty set X with a binary operation $*$ and a constant 0 satisfying the following conditions:

BCI-1 $((x*y)*(x*z))*(z*y)=0$,

BCI-2 $(x*(x*y))*y=0$,

BCI-3 $x*x=0$,

BCI-4 $x*y=0$ and $y*x=0$ imply $x=y$,

for all $x,y,z\in X$.

A nonempty subset A of a BCI-algebra X is said to be a subalgebra if $x\in A$ and $y\in A$ imply $x*y\in A$.

An ideal of a BCI-algebra X is a subset I containing 0 such that if $x*y\in I$ and $y\in I$ then $x\in I$. It is a closed ideal of X if whenever $x\in I$ then so does $0*x$.

We note that every closed ideal is a subalgebra ([3]).

In a BCI-algebra X, the following identities hold:

(1) $x*0=x$.

(2) $(x*y)*z=(x*z)*y$.

(3) $0*(x*y)=(0*x)*(0*y)$.

For any BCI-algebra X and $x,y\in X$, denote

$$A(x,y)=\{z\in X|(z*x)*y=0\}.$$

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THEOREM 1. *If I is an ideal of a BCI-algebra X, then*

$$I = \bigcup_{x,y \in I} A(x,y).$$

Proof. Let I be an ideal of a BCI-algebra X. If $z \in I$ then since $(z * 0) * z = (z * z) * 0 = 0 * 0 = 0$, we have $z \in A(0,z)$. Hence

$$I \subseteq \bigcup_{z \in I} A(0,z) \subseteq \bigcup_{x,y \in I} A(x,y).$$

Let $z \in \bigcup_{x,y \in I} A(x,y)$. Then there exist $a, b \in I$ such that $z \in A(a,b)$, so that $(z * a) * b = 0$. Since I is an ideal, it follows that $z \in I$. Thus $\bigcup_{x,y \in I} A(x,y) \subseteq I$, and consequently $I = \bigcup_{x,y \in I} A(x,y)$.

COROLLARY 2. *If I is an ideal of a BCI-algebra X, then*

$$I = \bigcup_{x \in I} A(0,x).$$

Proof. By Theorem 1 we have that $\bigcup_{x \in I} A(0,x) \subseteq \bigcup_{x,y \in I} A(x,y) = I$. If $x \in I$ then $x \in A(0,x)$ because $(x * 0) * x = 0$. Hence $I \subseteq \bigcup_{x \in I} A(0,x)$. This completes the proof.

THEOREM 3. *Let I be a subset of a BCI-algebra X such that $0 \in I$ and $I = \bigcup_{x,y \in I} A(x,y)$. Then I is an ideal of X.*

Proof. Let $x * y, y \in I = \bigcup_{x,y \in I} A(x,y)$. It follows from BCI-2 that $x \in A(x * y, y) \subseteq I$. Hence I is an ideal of X.

Combining Theorems 1 and 3, we have the following corollary.

COROLLARY 4. *Let X be a BCI-algebra and I a subset of X containing 0. Then I is an ideal of X if and only if $I = \bigcup_{x,y \in I} A(x,y)$.*

Now we give a characterization of closed ideals.

THEOREM 5. *Let I be a subset of a BCI-algebra X. Then I is a closed ideal of X if and only if it satisfies*

(i) $0 \in I$,

(ii) $x * z \in I$, $y * z \in I$ *and* $z \in I$ *imply* $x * y \in I$.

Proof. Let I be a closed ideal of X. Clearly $0 \in I$. Assume that $x * z, y * z, z \in I$. Since I is an ideal, therefore $x, y \in I$, which implies that $x * y \in I$ because I is a closed ideal and hence a subalgebra.

Conversely assume that I satisfies (i) and (ii). Let $x * y, y \in I$. Since $0 * 0, y * 0, 0 \in I$, by (ii) we have $0 * y \in I$. From (ii) again it follows that $x = x * 0 \in I$, so that I is an ideal of X. Now suppose $x \in I$. Noticing that $0 * 0, x * 0, 0 \in I$; then $0 * x \in I$ follows from (ii). This completes the proof.

THEOREM 6. *Let I be an ideal of a BCI-algebra X. The set*

$$I^0 = \{x \in I \,|\, 0 * x \in I\}$$

is the greatest closed ideal of X which is contained in I.

Proof. First we show that I^0 is an ideal of X. Clearly $0 \in I^0$. For any $x, y \in X$, if $x * y, y \in I^0$, then $0 * y \in I$ and

$$(0 * x) * (0 * y) = 0 * (x * y) \in I.$$

Since I is an ideal of X, it follows that $0 * x \in I$. Moreover since $I^0 \subseteq I$, therefore $x * y, y \in I^0 \subseteq I$ implies $x \in I$. Hence $x \in I^0$, and so I^0 is an ideal of X. If $x \in I^0$, then $0 * x \in I$. Since $(0 * (0 * x)) * x = 0$, it follows that $0 * (0 * x) \in I$. Hence $0 * x \in I^0$, which proves that I^0 is closed. Now assume that A is a closed ideal of X which is contained in I. Let $x \in A$. Then $0 * x \in A$. Since A is contained in I, therefore $x, 0 * x \in I$, and so $x \in I^0$. Thus $A \subseteq I^0$. Therefore I^0 is the greatest closed ideal which is contained in I.

DEFINITION 7. An ideal I of a BCI-algebra X is called an ignorable ideal of X if $I^0 = \{0\}$.

THEOREM 8. *Let I be an ideal of a BCI-algebra X. Then $I^g = (I - I^0) \cup \{0\}$ is an ignorable ideal of X.*

Proof. Let $x, y \in X$ be such that $x * y \in I^g$ and $y \in I^g$. If $y = 0$ then $x = x * 0 \in I^g$. Assume that $y \neq 0$. Clearly $x * y, y \in I$, which implies that $x \in I$. If $x \in I^0 - \{0\}$, then $x \neq 0$ and $0 * x \in I$. Since $y \neq 0$, it follows from $y \in I^g$ that $y \in I - I^0$, so that $0 * y \notin I$. On the other hand since $((0 * y) * (0 * x)) * (x * y) = 0$ and since $x * y \in I$, we have that $(0 * y) * (0 * x) \in I$, so that $0 * y \in I$. This is a contradiction. Hence $x \notin I^0 - \{0\}$, i.e., $x \in I^g$. This proves that I^g is an ideal of X. Now we show that $(I^g)^0 = \{0\}$. If $x \in (I^g)^0$, then $x \in I^g$ and $0 * x \in I^g$. From $x \in I^g$ it follows that $x = 0$ or $x \in I - I^0$. If $x \in I - I^0$ then $0 * x \notin I$, which is a contradiction. Thus $x = 0$. This completes the proof.

The following corollary is obvious.

COROLLARY 9. *Let I be an ideal of a BCI-algebra X. Then*

$$I^0 \cup I^g = I \text{ and } I^0 \cap I^g = \{0\}.$$

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