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NORMAL BCI/BCK-ALGEBRAS

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In 1966, Iséki [2] introduced the notion of BCI-algebras which is a generalization of BCK-algebras. Lei and Xi [3] discussed a new class of BCI-algebra, which is called a p-semisimple BCI-algebra. For p-semisimple BCI-algebras, a subalgebra is an ideal. But a subalgebra of an arbitrary BCI/BCK-algebra is not necessarily an ideal.

In this note, a BCI/BCK-algebra that every subalgebra is an ideal is called a normal BCI/BCK-algebra, and we give characterizations of normal BCI/BCK-algebras. Moreover we give a positive answer to the problem which is posed in [4].

Let us review some definitions and results.

By a BCI-algebra we mean an abstract algebra $(X;*,0)$ of type (2,0) satisfying the following conditions:

BCI-1 $((x*y)*(x*z))*(z*y)=0$,

BCI-2 $(x*(x*y))*y=0$,

BCI-3 $x*x=0$,

BCI-4 $x*y=0$ and $y*x=0$ imply $x=y$,

for all $x,y,z\in X$.

A BCI-algebra X satisfying

BCK-5 $0*x=0$ for all $x\in X$

is said to be a BCK-algebra.

In a BCI/BCK-algebra X we can define a partial ordering $\leq$ by putting $x\leq y$ if and only if $x*y=0$.

A nonempty subset A of a BCI/BCK-algebra X is said to be a subalgebra if it satisfies

(i) $x\in A$ and $y\in A$ imply $x*y\in A$.

A subset I of a BCI/BCK-algebra X is said to be an ideal if it satisfies

(ii) $0\in I$,

(iii) $x*y\in I$ and $y\in I$ imply $x\in I$.

An ideal I of a BCI-algebra X is said to be closed if it satisfies

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(iv) $x \in I$ implies $0 * x \in I$.

In a BCI-algebra X, the following identities hold:

(1) $x * 0 = x$.
(2) $(x * y) * z = (x * z) * y$.
(3) $0 * (x * y) = (0 * x) * (0 * y)$.
(4) $x * (x * (x * y)) = x * y$.

A BCI-algebra X is said to be p-semisimple if $X_+ = \{0\}$, where $X_+ = \{x \in X : 0 * x = 0\}$ which is called the BCK-part of X.

Let X be a BCI-algebra. An element a of X is said to be an atom if $x * a = 0$ implies $x = a$ for every $x \in X$. Denote the set of all atoms of X by $L(X)$. For all $a \in L(X)$, $V(a) = \{x \in X : a \leq x\}$ is said to be a branch of X. Clearly $V(0) = X_+$.

PROPOSITION 1 ([5]). *Let X be a BCI-algebra and let $a, b \in L(X)$. Then*

(5) *$x \in L(X)$ if and only if $x = y * (y * x)$ for all $y \in X$.*
(6) *$a * x \in L(X)$ for all $x \in X$.*
(7) *$x * y \in V(a * b)$ for $x \in V(a)$ and $y \in V(b)$.*
(8) *$L(X)$ is a p-semisimple BCI-algebra.*
(9) *X is p-semisimple if and only if $X = L(X)$.*

Let A be a nonempty subset of a BCI-algebra X. Denote $\{x \in A : x = 0 * (0 * x)\}$ by $L(A)$. Obviously $L(A) = A \cap L(X)$.

PROPOSITION 2. *Let A be a subalgebra of a BCI-algebra X. Then $L(A)$ is a closed ideal of $L(X)$.*

Proof. The proof is easy, and we omit the proof.

PROPOSITION 3 ([1]). *Let I be a subset of a p-semisimple BCI-algebra X. Then I is a closed ideal if and only if I is a subalgebra.*

A BCI-algebra X is said to be a KL-product BCI-algebra([6]) if there exist a BCK-algebra Y and a p-semisimple BCI-algebra Z such that $X \cong Y \times Z$.

PROPOSITION 4 ([6]). *Let X be a BCI-algebra. Then the following are equivalent:*

(10) *X is of KL-product.*
(11) *$L(X)$ is an ideal of X.*
(12) *$X \cong X_+ \times L(X)$.*

PROPOSITION 5 ([4]). *Let X be a BCI-algebra. Then the following are equivalent:*

(10) *X is of KL-product.*

(13) $x = (x * a) * (0 * a)$ *for* $x \in X$ *and* $a \in L(X)$.

(14) $(x * a) * (y * b) = (x * y) * (a * b)$ *for* $x, y \in X$ *and* $a, b \in L(X)$.

DEFINITION 6. A BCI/BCK-algebra X is said to be normal if every subalgebra of X is an ideal.

EXAMPLE 7. (a) Every p-semisimple BCI-algebra is normal.

(b) If every nonzero element in a BCK-algebra X is an atom, then X is a normal BCK-algebra.

(c) If a BCI-algebra X is the direct product of X_+ and $L(X)$, and if every nonzero element of X_+ is an atom, then X is a normal BCI-algebra.

The following proposition is obvious.

PROPOSITION 8. *Let X be a normal BCI-algebra. Then every subalgebra of X is a normal BCI-algebra. Particularly, X_+ is a normal BCK-algebra.*

THEOREM 9. *A BCK-algebra X is normal if and only if it satisfies*

(15) $x * y = x$ *for* $x \neq y$.

Proof. Suppose that X is a normal nontrivial BCK-algebra. Then for all $x \in X$, $\{0, x\}$ is an ideal of X, as $\{0, x\}$ is a subalgebra of X. Let $x, y \in X$ be such that $x \neq y$. Since $x * y \leq x$, we have that $x * y \in \{0, x\}$. Hence $x * y = 0$ or $x * y = x$. If $x * y = 0$, then $x \leq y$. It follows that $x \in \{0, y\}$ so that $x = 0$. Thus X satisfies (15).

Conversely suppose that X satisfies (15). Let A be an arbitrary subalgebra of X. Let $x * y, y \in A$. If $x \neq y$, then $x = x * y$ by (15). Hence $x \in A$. If $x = y$ then clearly $x \in A$. Thus A is an ideal of X. This completes the proof.

THEOREM 10. *A BCI-algebra X is normal if and only if X is of KL-product and X_+ satisfies the condition* (15).

Proof. Suppose that X is a normal BCI-algebra. Then $L(X)$ is an ideal of X, as $L(X)$ is a subalgebra of X. It follows from Proposition 4 that X is of KL-product. By Proposition 8 and Theorem 9, we have that X_+ satisfies (15).

Conversely assume that X is of KL-product and X_+ satisfies (15). Let A be an arbitrary subalgebra of X. We know by Proposition 2 that $L(A)$ is a closed ideal of $L(X)$. Now let $x * y, y \in A$. Then we have that $0 * y \in A$ and $(0 * x) * (0 * y) = 0 * (x * y) \in A$. By means of (4), we conclude that $0 * y, (0 * x) * (0 * y) \in L(A)$. Thus $0 * x \in L(A) \subseteq A$. If $x * (0 * (0 * x)) = y * (0 * (0 * y))$, then by (4) and (13)

$$x = (x * (0 * (0 * x))) * (0 * x) = (y * (0 * (0 * y))) * (0 * x) \in A.$$

If $x * (0 * (0 * x)) \neq y * (0 * (0 * y))$, then by (4), (13), (14) and (15)

$$\begin{aligned} x &= (x * (0 * (0 * x))) * (0 * x) \\ &= ((x * (0 * (0 * x))) * (y * (0 * (0 * y)))) * (0 * x) \\ &= ((x * y) * (0 * (0 * (x * y)))) * (0 * x) \in A. \end{aligned}$$

Therefore A is an ideal of X, and hence X is normal. The proof is complete.

COROLLARY 11. *Let X be a BCI-algebra. Then X is normal if and only if $X \cong X_+ \times L(X)$ and X_+ is a normal BCK-algebra.*

We recall that a mapping $f : X \to Y$ of BCI-algebras is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

DEFINITION 12 ([4]). Let X be a BCI-algebra. The mapping $p : X \to X$ is defined by putting $p(x) = x * a$ for all $x \in X$, where $a = 0 * (0 * x) \in L(X)$, that is, $p(x) = x * (0 * (0 * x))$.

PROPOSITION 13. *Let X be a BCI-algebra. Then $x \in L(X)$ if and only if $p(x) = 0$.*

Proof. Let $x \in L(X)$. Then $0 * (0 * x) = x$. Hence

$$p(x) = x * (0 * (0 * x)) = 0.$$

Conversely assume that $p(x) = 0$, i.e., $x * (0 * (0 * x)) = 0$. Then $x \leq 0 * (0 * x)$. Combining BCI-2, we know that $x = 0 * (0 * x)$. Hence $x \in L(X)$. This completes the proof.

Combining Proposition 13 and [5; Theorem 1], we have the following corollary.

COROLLARY 14. *Let X be a BCI-algebra. Then for all x, y, u, z of X, the following conditions are equivalent:*

(a) $p(x) = 0$.
(b) $x = z * (z * x)$.
(c) $(z * u) * (z * x) = x * u$.
(d) $x * (z * y) \leq y * (z * x)$.
(e) $(x * u) * (z * y) \leq (y * u) * (z * x)$.
(f) $(0 * z) * (0 * x) = x * z$.
(g) $0 * (0 * x) = x$.
(h) $0 * (z * x) = x * z$.
(i) $0 * (0 * (x * z)) = x * z$.
(j) $z * (z * (x * u)) = x * u$.

THEOREM 15. *Let X be a BCI-algebra. If p is an endomorphism on X, then X is a KL-product BCI-algebra.*

Proof. Suppose that p is an endomorphism on X. Let $x * y, y \in L(X)$. Then by (1) and Proposition 13, we have

$$p(x) = p(x) * 0 = p(x) * p(y) = p(x * y) = 0.$$

It follows from Proposition 13 that $x \in L(X)$, so that $L(X)$ is an ideal of X. Proposition 4 assures us that X is a KL-product BCI-algebra.

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