Comm. Korean Math. Soc. 9 (1994), No. 2, pp. 261–264

LOCALLY COMPLETE INTERSECTION IDEALS IN COHEN-MACAULAY LOCAL RINGS

MEE-KYOUNG KIM

Throughout this paper, all rings are assumed to be commutative with identity. By a local ring (R, m), we mean a Noetherian ring R which has the unique maximal ideal m. By $\dim(R)$ we always mean the Krull dimension of R. Let I be an ideal in a ring R and t an indeterminate over R. Then the Rees algebra $R[It]$ is defined to be

$$R[It] = R \oplus It \oplus I^2t^2 \oplus \cdots \quad .$$

Let (R, m) be a local ring and I an ideal of R. An ideal J contained in I is called a *reduction* of I if $JI^n = I^{n+1}$ for some integer $n \geq 0$. A reduction J of I is called a *minimal reduction* of I if J is minimal with respect to being a reduction of I. The *analytic spread* of I, denoted by $l(I)$, is defined to be $\dim\big(R[It]/mR[It]\big)$. In [5], it is shown that $ht(I) \leq l(I) \leq \dim(R)$. An ideal I is called *equimultiple* if $l(I) = ht(I)$. If R/m is an infinite field, then $l(I)$ is the least number of elements generating a reduction of I ([5]). We will use notation $\lambda_R(M)$ (or simply $\lambda(M)$) to denote the length of M as an R-module and $\mu(I)$ to denote the number of elements in a minimal basis of an ideal I of a local ring (R, m) (i.e., $\mu(I) = \lambda(I/mI)$). An ideal I is called *complete intersection* if $ht(I) = \mu(I)$. An ideal I is called a *locally complete intersection* if IR_p is a complete intersection for all $p \in Ass_R(R/I)$. Let $e(R)$ denote the multiplicity of R relative to m. As a general reference, we refer the reader to [4] for any unexplained notation or terminology.

In this paper, we show that an equimultiple ideal I is generated by a regular sequence (i.e., a complete intersection) if I is a locally complete intersection in a Cohen-Macaulay local ring (R, m).

Received August 23, 1993.

PROPOSITION 1. *Let (R, m) be a Cohen-Macaulay local ring with an infinite residue field and I an ideal in R. Suppose that I is a locally complete intersection. Then I has no embedded primes.*

Proof. Since I is a locally complete intersection, we have that

$$ht(IR_p) = \mu(IR_p)$$

for all $p \in Ass_R(R/I)$. Hence IR_p has no embedded primes in R_p, since R_p is a Cohen-Macaulay local ring for all $p \in Ass_R(R/I)$.

Claim : I has no embedded primes.

Suppose that I has an embedded prime. That is, there exists a prime ideal q in R such that $I \subseteqq q \subsetneqq p$. Since IR_p is unmixed for all $p \in Ass_R(R/I)$ ([4], Theorem 17.6), we have that

$$ht(qR_p) = ht(pR_p)\,.$$

Hence we have that $q = p$ because $q \subsetneqq p$. It's a contradiction and this completes the proof of our assertion.

REMARK. Let R_p be a Cohen-Macaulay local ring for all $p \in Ass_R(R/I)$. If IR_p is unmixed for all $p \in Ass_R(R/I)$, then I has no embedded primes.

LEMMA 2. *Let I and J be ideals of a Noetherian ring R. If $JR_p \subseteqq IR_p$ for every $p \in Ass_R(R/I)$, then $J \subseteqq I$.*

Proof. Let $I = q_1 \cap q_2 \cap \cdots \cap q_r$ be a minimal primary decomposition of I, where $\sqrt{q_i} = p_i$ for $i = 1, 2, \ldots, r$. Then we have that $Ass_R(R/I) = \{p_1, p_2, \ldots, p_r\}$. Suppose that $J \nsubseteqq I$. Then there exists an element x in J such that $x \notin I$. Hence we have that $x \notin q_i$ for some $1 \leq i \leq r$. So $(q_i : x)$ is p_i-primary and $\sqrt{(q_i : x)} = p_i$ ([1], Lemma 4.4.). We get by the hypothesis that

$$\frac{x}{1} \in IR_{p_i}\,.$$

This allows us to express

$$\frac{x}{1} = \frac{a}{s}$$

with $a \in I$ and $s \notin p_i$. In this situation, we see that $tsx \in I$ for some $t \notin p_i$. Hence we have that

$$ts \in (I : x) \subseteqq (q_i : x).$$

Therefore we see that $ts \in p_i$, which is a contradiction. This completes the proof of our assertion.

THEOREM 3. *Let* (R, m) *be a Cohen-Macaulay local ring with an infinite residue field and let* I *be an equimultiple ideal of* $ht(I) = r$. *Suppose that* I *is a locally complete intersection. Then* I *is generated by a regular sequence on* R.

Proof. Since I is an equimultiple ideal of $ht(I) = r$ with $|R/m| = \infty$, there exists a minimal reduction $J = (a_1, a_2, \dots, a_r)$ of I. So we have that $ht(J) = \mu(J) = r$, since $\sqrt{I} = \sqrt{J}$. Hence J is generated by a regular sequence on R, since (R, m) is a Cohen-Macaulay local ring.

Claim : $J = I$.

$\subseteqq$: It is obvious.

$\supseteqq$: Since J is complete intersection in a Cohen-Macaulay local ring (R, m), J is unmixed, i.e.,

$$Ass_R(R/J) = \mathrm{Min}(J).$$

I has no embedded primes, by Proposition 1, i.e.,

$$Ass_R(R/I) = \mathrm{Min}(I).$$

Notice that $\mathrm{Min}(J) = \mathrm{Min}(I)$, because J is a reduction of I ([5]). Consequently

$$Ass_R(R/J) = \mathrm{Min}(J) = \mathrm{Min}(I) = Ass_R(R/I).$$

Condition of a locally complete intersection tells us ([5], §4, Theorem 2) that IR_p has no proper reduction for all $p \in Ass_R(R/I)$. Hence we get that

$$JR_p = IR_p \qquad \text{for all} \quad p \in Ass_R(R/I).$$

Thus we see that $J \supseteqq I$ by Lemma 2. The proof of claim is complete and this completes the proof of our assertion.

The following example shows that Theorem 3 is false for an equimultiple and prime ideal $I = p$ which does not satisfy the condition of a locally complete intersection.

. EXAMPLE 4. Let

$$R = k[[X, Y, Z, W]]/(Z^2 - W^5, Y^2 - XZ)$$
$$= k[[x, y, z, w]]$$

and $p = (y, z, w)$.
We have $wp^3 = p^4$, hence $l(p) = ht(p) = 1$. Furthermore $R/p \simeq k[[x]]$ is regular. Therefore by [3] we get equimultiplicity : $e(R) = e(R_p)$. Surely $e(R) > 1$, hence $e(R_p) \geq 2$, i.e., R_p is not regular. But R is a 2-dimensional Cohen-Macaulay local ring. Now in this case p is not generated by a regular sequence.

References

1. M. Atiyah and I. MacDonald, *Introduction to Commutative Algebra*, Addison Wesley, 1969.
2. M. Herrmann and S. Ikeda, *Remarks on lifting of Cohen-Macaulay property*, Nagoya Math. J. **92** (1983), 121–132.
3. M. Herrmann and U. Orbanz, *Faserdimension von Aufblasungen lokaler Rings und Äquimultiplizität*, J. Math. Kyoto Univ. **20** (1980), 651–659.
4. H. Matsumura, *Commutative ring Theory, Cambridge Studies in Advanced Math. 8*, Cambridge Univ. Press, 1986.
5. D. G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge Phil. Soc. **50** (1954), 145–158.

Department of Mathematics
Sung Kyun Kwan University
Suwon 440-746, Korea