The Journal of the Korean Society of
Coastal and Ocean Engineers
Vol. 6, No. 3, pp. 275~280, September 1994

Re-derivation of Addded Mass Coefficient of Circular Cylinder near Bottom Boundary
바닥경계 가까이 있는 원형실린더의 부가질량계수의 재유도

Chong Keun Pyun* and Chang Kun Park**
편종근* · 박창근**

Abstract□The analytic solution of the forces acting on a horizontal circular cylinder affected by the bottom boundary is re-derived using the complex potential. The reason of the re-derivation of the analytic solution is such that it is found that the analytic solution of Yamamoto *et al.* (1974) does not simulate the behavior of the added mass coefficient accurately. The re-derived formula of the added mass coefficient C_M is different from that of Yamamoto *et al.* (1974), while the re-derived formula of C_M simulates the tendency of the behavior of the added mass coefficient successfully.

要 旨: 바닥영향을 받는 수평원형 실린더에 작용하는 힘에 대한 해석해를 복소포텐셜을 이용하여 재유도하였다. 본 연구에서 해석해를 재유도한 이유는 Yamamoto 등(1974)이 유도한 해석해는 부가질량계수가 거동하는 경향을 정확하게 설명하지 못하고있기 때문이다. 재유도한 부가질량계수 C_M은 Yamamoto 등(1974)의 C_M과 다르며, 또한 부가질량계수가 거동하는 경향을 만족하게 설명하고 있음을 알 수 있었다.

1. INTRODUCTION

A circular cylinder is one of the most efficient forms in the marine structures, because it has the cross section supporting the largest external force per unit surface area, and because the external forces acting on a circular cylinder can be analyzed simply because of its symmetry. The cylinder near the sea bottom may become a pipeline. Therefore, the analysis of the forces acting on the pipeline follows naturally that the forces acting on a horizontal cylinder are analyzed by considering the effects of the boundary conditions.

In a two-dimensional irrotational flow of inviscid incompressible fluid, the force(F) per unit width acting on a circular cylinder consists of the inertia force(F_x) parallel to the direction of the flow and the lift force(F_y) perpendicular to the direction of the flow, whose mathematical form may be expressed as follows:

$$F=F_x+F_y=\rho C_I A\frac{\partial U}{\partial t}+0.5\rho C_L DU^2 \qquad (1)$$

where ρ is the density of water, A is the cross-sectional area of the cylinder, D is the diameter of the cylinder, U is the horizontal fluid particle velocity, t is the time, C_I is the inertia force coefficient, C_L is the lift force coefficient. The relation of C_L with C_M can be expressed as $C_L=1+C_M$ where C_M is the added mass coefficient. In the case of this flow, the drag force does not exist.

An analytic solution of Eq. (1) is useful to understand the behavior of the pipeline near the bottom boundary in the sea, and may give some criteria for the design of a pipeline. But the analytic solution cannot be obtained simply because Eq. (1) is

*서울대학교 토목공학과 (Department of Civil Engineering, Seoul National University, Seoul, 151-741, Korea)
**명지대학교 토목공학과 (Department of Civil Engineering, Myungji University, Yongin-Gun, Kyungki-Do, 449-728, Korea)

subject to both the free surface boundary condition and the bottom boundary condition. Havelock (1936) developed an analytic solution for an uniform steady horizontal flow, where the effect of the free surface on the forces acting on a cylinder submerged in a flow is negligible if the submergence is greater than about four cylinder diameters. Ogilvie (1963) derived the forces acting on a cylinder which is oscillating near the free surface. If a cylinder is submerged 3 or 4 diameters, the added mass coefficient resulting from the effect of the free surface may be negligible. Yamamoto *et al.* (1974) reported that for a cylinder submerged as little as one diameter the forces resulting from the surface effect are small. Therefore, the effect of the free surface boundary on the forces may be negligible for a cylinder near the sea-bottom.

In this study re-derivation of an analytic solution of the forces acting on a circular cylinder is performed based on the procedures of Yamamoto *et al.* (1974), where the bottom boundary condition is considered and the free surface boundary condition is neglected. The re-derived solution is compared with the solutions of Yamamoto *et al.* (1974). and The reason of the re-derivation of the analytic solution is as follows: the analytic solutions of Yamamoto *et al.* (1974), and Nath and Yamamoto (1974) do not simulate the tendency of the added mass coefficient accurately.

2. ADDED MASS COEFFICIENT

The pressure acting on a fixed cylinder consists of P_{inf} and P^*, where P_{inf} is the pressure resulting from the uniform flow far from the cylinder and P^* is the pressure resulting from the distorted flow region near the cylinder. Therefore, the pressure acting on a fixed cylinder in an uniform flow with the velocity U is $P_{inf}+P^*$, while the pressure acting on an oscillating cylinder in a calm water is P^*. The P_{inf}-induced force per unit width acting on a cylinder is expressed as $\rho \cdot A \cdot \partial U/\partial t$, which is known as the Froud-Krylov force. On the other hand, because a distribution of P^* on the surface of a cylinder may not be calculated easily, the concept of the addes mass coefficient is introduced. If it is assumed that P^* accelerates fluid particles surrounding the cylinder with acceleration $\partial U/\partial t$, the amount of accelerated fluid particles is defined as the added mass. Therefore, the added mass coefficient C_M can be defined as follows:

$$C_M = \frac{\left[\begin{array}{c}\text{Assumed amount of accelerated fluid} \\ \text{with } \partial U/\partial t \text{ resulting from } P^*\end{array}\right]}{\left[\begin{array}{c}\text{Amount of fluid corresponding to} \\ \text{volume of cylinder}\end{array}\right]} \tag{2}$$

Therefore, the inertia force coefficient in Eq. (1) may be expressed as $C_I=1+C_M$. $C_M=0.5$ for a sphere in an uniform flow and $C_M=1.0$ for a cylinder based on the analysis of the potential flow with neglecting the boundary conditions, thus C_M may be greater than 1 for a cylinder affected by the bottom boundary.

3. RE-DERIVATION OF ANALYTIC SOLUTION

In a two-dimensional irrotational flow of inviscid incompressible fluid, consider a doublet(strength$=\mu$, $\mu=\mu_o$) near the bottom boundary apart from iS_o ($S_o=S$) which moves with the horizontal velocity U as shown in Fig. 1,
where $i=(-1)^{1/2}$, $a(=D/2)$ is the radius of the cylin-

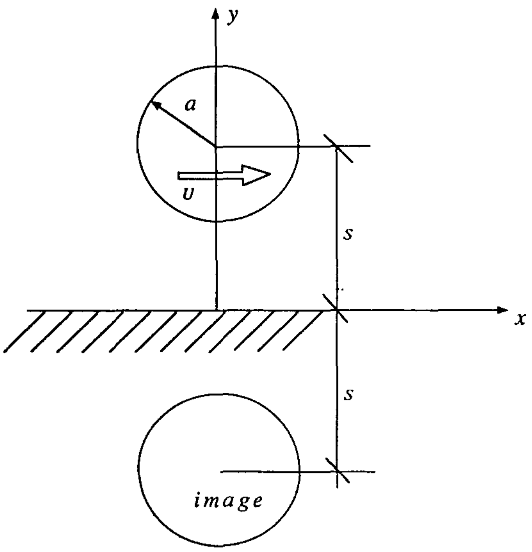

Fig. 1. Sketch of Doublet affected by Bottom Boundary.

der, and S(>a) is the distance between the bottom and the center of the doublet. Since the bottom boundary condition($\partial\Phi/\partial y=0$ at $y=0$; Φ is the velocity potential) does not meet because of the existing original doublet, the bottom condition can be satisfied by setting an image doublet with the same strength($=\mu_o$) at the image point(0, $-iS_o$). Therefore, the complex potential W_0 for above two doublets may become

$$W_o=\mu_o\left(\frac{1}{z-iS_o}+\frac{1}{z+iS_o}\right),\ \mu_o=U\cdot a^2 \quad (3)$$

where $z=x+iy$.

The original circular cylinder is distorted because of the image doublet whose function is to satisfy the bottom boundary condition. To make the distorted cylinder circular, another doublet with the strength μ_1 is set at the point (0, iS_1) within the original circular cylinder. An image doublet with strength μ_1 is set at the image point (0, $-iS_1$) to satisfy the bottom boundary condition. After the above procedures are repeated infinitely, the shape of the original cylinder may become circular and the straight bottom boundary may be obtained. Therefore, the complex potential W describing both the original circular cylinder and the straight bottom boundary can be expressed in the form of series:

$$W=\Phi+i\Psi=\sum_{k=0}^{\infty}\mu_k\left(\frac{1}{z-iS_k}+\frac{1}{z+iS_k}\right) \quad (4)$$

where Φ is the velocity potential, and Ψ is the stream function. In the above complex potential, the position(S_k) and the strength(μ_k) of the k-th doublet within the original doublet must be specified. Therefore, it is required to calculate the position(S_k) and the strength(μ_k) of the k-th doublet whose function is to compensate the distorted original doublet resulting from the (k−1)-th image doublet. The complex potential W′ for these two doublets may become

$$W'=\mu_{k-1}\frac{1}{z+iS_{k-1}}+\mu_k\frac{1}{z-iS_k} \quad (5)$$

In order to meet the condition that the original doublet has to become circular, the following condition may be satisfied:

$$\frac{\partial\Phi'}{\partial r}=0 \text{ on } r=a \quad (6)$$

where Φ' is the velocity potential in $W'=\Phi'+i\Psi'$. By using the relation of $z=r(\cos\theta+i\sin\theta)$ and by substituting Eq. (5) into Eq. (6), the following recurrent expressions for S_k and μ_k may be obtained, respectively:

$$S_k=S-\frac{a^2}{S+S_{k-1}} \quad (7)$$

$$\mu_k=\mu_{k-1}\left(\frac{a}{S+S_{k-1}}\right)^2 \quad (8)$$

For the sake of efficient calculation, a parameter q_k is defined as follows:

$$q_k\equiv\frac{S-S_k}{a}=\frac{a}{S+S_{k-1}}=\frac{1}{2S/a-q_{k-1}},\ q_o=0 \quad (9)$$

where $S=S_o$, and the strength μ_k of the k-th doublet can be obtained.

$$\mu_k=\mu_{k-1}\cdot q_k^2=\mu_o(q_1\cdot q_2\cdots q_k)^2=Ua^2(q_1\cdot q_2\cdots q_k)^2$$
$$=Ua^2\cdot m_k;\ m_k=(q_1\cdot q_2\cdots q_k)^2,\ m_o=1 \quad (10)$$

By substituting Eq. (9) and Eq. (10) into Eq. (4), the interesting complex potential can be obtained.

$$W=Ua^2\sum_{k=0}^{\infty}m_k\left(\frac{1}{z-i(S-aq_k)}+\frac{1}{z+i(S-aq_k)}\right) \quad (11)$$

$$q_k=\frac{1}{2S/a-q_{k-1}},\quad q_o=0;$$

$$m_k=(q_1\cdot q_2\cdots q_k)^2,\ m_o=1$$

The forces acting on the surface of a circular cylinder in an unsteady irrotational flow may be expressed as the Blasius theorem:

$$F=F_x-iF_y=-i\rho\oint\frac{\partial\Phi}{\partial t}d\bar{z}+i\frac{1}{2}\rho\oint\left(\frac{\partial W}{\partial z}\right)^2dz \quad (12)$$

where $z=x+iy$ and $\bar{z}=x-iy$. Before performing the integration of the first term of the right hand side in Eq. (12), it is necessary to calculate $\partial\Phi/\partial t$ by using Eq. (11).

$$\frac{\partial\Phi}{\partial t}=\frac{1}{2}\left(\frac{\partial W}{\partial t}+\frac{\partial\bar{W}}{\partial t}\right)$$

$$=\frac{1}{2}\sum_{k=0}^{\infty}\frac{\partial\mu_k}{\partial t}\left(\frac{1}{z-i(S-aq_k)}+\frac{1}{z+i(S-aq_k)}+\frac{1}{\bar{z}+i(S-aq_k)}+\frac{1}{\bar{z}-i(S-aq_k)}\right) \tag{13}$$

Because the integration of Eq. (12) is performed on the surface of the original doublet, it is necessary to use the relation of $z-iS=a(\cos\theta+i\sin\theta)$ and $\bar{z}-iS=a(\cos\theta+i\sin\theta)$ in Eq. (13), where a is the radius of the original doublet. By using the following relation,

$$\int_0^{\pi}\frac{\mathrm{con}(nx)\cdot dx}{1-2a\cos x+a^2}=\begin{cases}\pi a^n/(1-a^2) & \text{when } a^2<1\\ \pi/[(1-a^2)a^n] & \text{when } a^2>1\end{cases} \tag{14}$$

the first term of Eq. (12) can be calculated as follows:

$$-i\rho\oint\frac{\partial\Phi}{\partial t}d\bar{z}=-\pi\rho\sum_{k=0}^{\infty}\frac{\partial\mu_k}{\partial t}\left(1+\frac{1}{(2S/a-q_j)^2}\right)$$

$$=-\pi\rho a^2\frac{\partial U}{\partial t}\sum_{k=0}^{\infty}(1+q_{j+1}{}^2)m_k \tag{15}$$

which is equal to F_x in Eq. (12). Since $F_x=\rho C_M\pi a^2\,\partial U/\partial t$ in the oscillating cylinder(instead of C_I, reference Eq. (1)), the added mass coefficient C_M can be obtained by comparison of Eq. (15).

$$C_M=\sum_{k=0}^{\infty}(1+q_{k+1}{}^2)m_k=1+2\sum_{k=1}^{\infty}m_k \tag{16}$$

In order to integrate the second term of the right hand side in Eq. (12), it is necessary to calculate $(\partial W/\partial t)^2$ by using Eq. (11) as follows:

$$\left(\frac{\partial W}{\partial t}\right)^2=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\mu_j\mu_k\frac{4(z^2-S_k{}^2)(z^2-S_j{}^2)}{(z^2-iS_k{}^2)(z^2+iS_k{}^2)(z^2-iS_j{}^2)(z^2+iS_j{}^2)} \tag{17}$$

and it is useful to employ the residue theorem in order to perform the integration. If an analytic function has many singular points $z_j(j=1,2,\cdots,n)$ inside a closed path, then

$$\oint f(z)dz=2\pi i\sum_{j=1}^{n}\operatorname*{Res}_{z=z_j}f(z) \tag{18}$$

If f(z) has a pole of m-th order at $z=z_0$, the residue is given as follows:

$$\operatorname*{Res}_{z=z_0}f(z)=\frac{1}{(m-1)!}\lim_{z\to z_0}\left(\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\right) \tag{19}$$

Therefore, by using Eq. (18) and Eq. (19) the second term of Eq. (12) can be calculated:

$$i\frac{1}{2}\rho\oint\left(\frac{\partial W}{\partial z}\right)^2 dz=i4\pi\rho U^2a^4\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{m_k\cdot m_j}{(2S-aq_k-aq_j)^3} \tag{20}$$

which is equal to $-F_y$. Since $F_y=0.5\rho C_L\,D\,U^2(D=2a)$ in Eq. (12), the lift force coefficient C_L can be obtained by comparison of Eq. (20).

$$C_L=-4\pi\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{m_j\ m_k}{\left(\frac{2S}{a}-q_j-q_k\right)^3} \tag{21}$$

On the other hand, Yamamoto *et al.* (1974) derived an analytic solution of the forces acting on a circular cylinder near the bottom boundary, where the lift force coefficient C_L and the inertia force coefficient C_M are expressed as follows:

$$C_L=-4\pi\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{m_j\ m_k}{\left(\frac{2S}{a}-q_j-q_k\right)^3};\ S>a \tag{22}$$

$$C_M=1+2\sum_{j=1}^{\infty}m_j{}^2 \tag{23}$$

The lift force coefficient(C_L) calculated in this study is the same as that of Yamamoto *et al.* (1974), while the added mass coefficient(C_M) is different from that of Yamamoto *et al.* (1974). The calculated C_L and C_M are shown in Fig. 2 and Fig. 3, respectively. The 900 image doublets were used to calculate C_L in Eq. (21) and C_M in both Eq. (16) and Eq. (23). The lift force calculated in the potential flow results from the difference of the pressure acting on the surface of the cylinder, and is proportional to U^2. C_L in Fig. 2 successfully simulates the tendency of the behavior of the lift force coefficient on a circular cylinder submerged in an ideal flow. When $e/D=1$ where D is the diameter of the cylinder and e $(=S-D/2$ in Fig. 1) is the distance between the bottom boundary and the lowest surface of the cylinder, the lift force coefficient $C_L=$

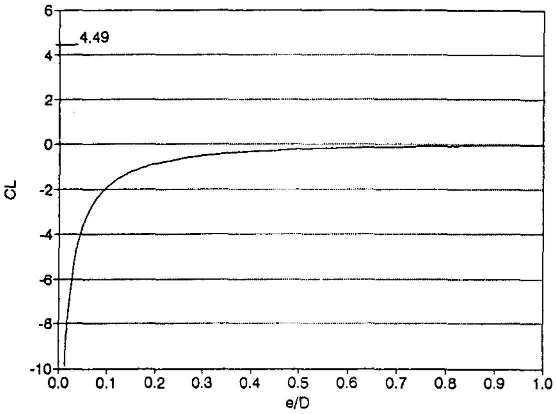

Fig. 2. Lift Force Coefficient (C_L).

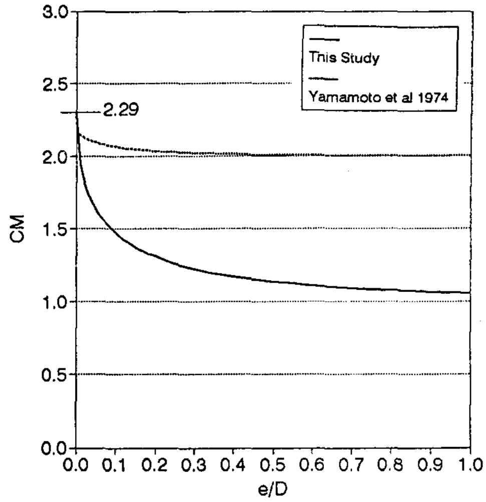

Fig. 3. Added Mass Coefficient (C_M).

$-0.06186 \fallingdotseq 0$ can be calculated. Therefore, the effect of the bottom boundary may be neglected in calculating the lift force when $e/D \geq 1$. As e/D decreases, C_L increases because of the increasing velocity between the cylinder and the bottom boundary, which means the existance of relatively lower pressure distribution on the lower surface of the cylinder. Since C_L is calculated as negative in the above case, the direction of the lift force is downward. When $e/D = 0$, that is, the cylinder is located on the bottom boundary, $C_L = +4.49 (= (\pi^3 + 3\pi)/9)$ can be calculated based on von Muller (1929)'s analytic solution, which means that the direction of the lift force is upward because there is no flow between the cylinder and the bottom boundary.

The inertia force is proportional to the acceleration $\partial U/\partial t$, and the added mass coefficient C_M is shown in Fig. 3. As e/D decreases, C_M increases because of the bottom boundary (in this case the acceleration increases between the cylinder and the boundary). When $e/D = 0$, that is, a cylinder is located on the bottom boundary, $C_M = 2.29$ can be obtained. When the cylinder is located at $e/D = 1$, $C_M = 1.05724 \fallingdotseq 1$, which is equivalent to the result obtained from the potential flow analysis of the forces acting on an oscillating circular cylinder where the effects of the bottom boundary are neglected. Therefore, the effects of the bottom boundary may be neglected in the case that $e/D \geq 1$. On the other hand, $C_M \fallingdotseq 2$ was calculated in the case of the solution of Yamamoto *et al.* (1974) when $e/D = 1$. This fact shows that the tendency of the behavior of Yamamoto *et al.* (1974)'s C_M is not appropriate because $C_M = 1$ is calculated approximately when $e/D = 1$.

4. CONCLUSION

The analytic solution of the forces acting on a cylinder affected by the bottom boundary is re-derived using the complex potential. Through the comparison of the re-derived analytic solution with the results of Yamamoto *et al.* (1974), the re-derived lift force coefficient C_L is identical with that of Yamamoto *et al.* (1974). But the re-derived added mass coefficient C_M is different from that of Yamamoto *et al.* (1974), while it simulates the tendency of the behavior of C_M successfully. If the effect of the bottom boundary is neglected in the analysis of the potential flow, $C_L = 0$ and $C_M = 1$ are calculated analytically for a circular cylinder. When $e/D \geq 1$ where D is the diameter of the cylinder and e is the distance between the bottom boundary and the lowest surface of the cylinder, the effects of the bottom boundary on the lift force and the inertia force can be neglected because the calculated C_L and C_M are approximately 0 and 1, respectively.

REFERENCES

Havelock, T.H., 1936. The forces on a cylinder submerged bodies, *Proceedings of the Royal Society, London*, **A157**: 526-534.

Nath, J.H. and Yamamoto, T., 1974. Forces from fluid flow around objects, *Proceedings of the 14th Coastal Engineering Conference*, Denmark, pp. 1808-1827.

Ogilvie, J.F., 1963. First and second order forces on a cylinder submerged under a free surface, *J. Fluid Mechanics*, **16**: 451-472.

Yamamoto, T., Nath, J.H. and Slotta, L.S., 1974. Wave forces on cylinders near plane boundary, *J. of the Waterways Harbors and Coastal Engineering Division*, **100** (WW4): 345-359.

von Muller, W., 1929. Systeme von Doppelquellen in der ebenen Stromung, insbesondere die Stromung um zwei Dreiszylinder, *Zeutschrift fur Angewandte Mathematik und Mechanik*, **9**: 200-213.